Journal of Number Theory 154 (2015) 375–379

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Function fields of class number one

Qibin Shen *, Shuhui Shi

Department of Mathematics, University of Rochester, Rochester, NY 14627 USA

A R T I C L E I N F O

Article history:
Received 4 December 2014
Received in revised form 21 February 2015
Accepted 22 February 2015
Available online 2 April 2015
Communicated by David Goss

Keywords:
Function field
Class number

A B S T R A C T

In 1975, J. Leitzel, M. Madan and C. Queen listed 7 function fields over finite fields (up to isomorphism) with positive genus and class number one. They claimed to prove that these were the only ones such. Recently, Claudio Stirpe found an 8th one! In this paper, we fix the argument in the former paper to show that this 8th example could have been found by their method and is the only one, so that the list is now complete.

1. Introduction

It is not yet known whether there are infinitely many number fields of class number one (let alone, real quadratic number fields of class number one, as Gauss conjectured). The classification of imaginary quadratic fields was completed by Heegner and Stark only in 1969, but it was only in 1983 through the works [G85] of Goldfeld and Gross–Zagier that it was established that the imaginary quadratic fields of given class number can be effectively classified.

In the case of (global) function fields (i.e., function fields over finite fields $\mathbb{F}_q$), there are no archimedean places at 'infinity', so there is no canonical ring of integers and its class group. In fact, there are several variants of class groups, see e.g., [T04, Chapter 1] for

* Corresponding author.
E-mail addresses: qshen4@ur.rochester.edu (Q. Shen), sshi10@ur.rochester.edu (S. Shi).

http://dx.doi.org/10.1016/j.jnt.2015.02.005
0022-314X/

more detailed discussion and references. The usual substitute, which we will use below, is the divisor class group of degree zero (or what is the same, the group of $\mathbb{F}_q$-rational points of the corresponding Jacobian).

All the genus zero function fields, namely the rational function fields $\mathbb{F}_q(t)$, one for each q, have class number one. MacRae [M71] classified class number one 'imaginary quadratic fields'. Also, we can, in fact, effectively determine all function fields of a given class number (even when q or the characteristic is not fixed), and we will quickly recall this below for the benefit of the reader. But even today, with the powerful computers, it is not so easy to do this even for class number one.

In M. Madan and C. Queen's paper [MQ72], it was shown that except for a possible exception of genus 4 function field with field of constants $\mathbb{F}_2$, there are only 7 class number one function fields of positive genus. Then J. Leitzel, M. Madan and C. Queen claimed to finish the classification in [LMQ75] by showing such exception does not exist. (A somewhat similar history to Heegner–Stark's proof of the non-existence of the last possible exception known since 1934!) But this was a mistake: recently Claudio Stirpe [S14] found an explicit 8th example, of the type ruled out.

As [S14] left the question open whether the example was the only counter-example, we went through the arguments of [LMQ75] and found exactly one more counter-example, the one given by Stirpe. After informing Stirpe of this, we were told by him that, jointly with Mercuri, he also had recently succeeded in proving the same result. The preprint [MSp14] has now appeared. We are submitting our independent work, cutting out unnecessary duplication with the work of Mercuri and Stirpe. It should be noted that [MSp14] offers two proofs, the second being essentially the same as ours.

2. Classifying function fields of a given class number

Let us recall some basic facts of the theory of global function fields (see e.g., [T04, Chapter 1] for references) for the benefit of the reader.

Let K be a function field with field of constants $\mathbb{F}_q$, of genus $g > 0$, class number h and zeta function $Z(t)$.

It is well-known through the works of Artin, Hasse and Weil that the class number h is equal to $P(1)$, where $P(t) = \prod_1^{2g}(t - \alpha_i)$ is the numerator of the zeta function. By Weil's theorem, which is the analog of Riemann hypothesis, the α_i's have absolute value $\sqrt{q}$. Hence

$$h \geq (\sqrt{q} - 1)^{2g}.$$

It follows that $h > 1$ if $q > 4$, and more generally, an upper bound on h implies an upper bound on g.

Let N_i be the number of $\mathbb{F}_{q^i}$-rational points of the projective non-singular curve corresponding to K, and let N denote the number of degree one primes for the constant