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The Mahler measure and its areal analog for totally positive algebraic integers

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ABSTRACT

Using the method of explicit auxiliary functions, we first improve the known lower bounds of the absolute Mahler measure of totally positive algebraic integers. In 2008, I. Pritsker defined a natural areal analog of the Mahler measure that we call *the Pritsker measure*. We study the spectrum of the absolute Pritsker measure for totally positive algebraic integers and find the four smallest points. Finally, we give inequalities involving the Mahler measure and the Pritsker measure of totally positive algebraic integers. The polynomials involved in the auxiliary functions are found by our recursive algorithm.

1. Introduction

The *Mahler measure* of a polynomial $P(z) = a_0 z^n + \ldots + a_n = a_0 \prod_{j=1}^{n} (z - \alpha_j) \in \mathbb{C}[X]$, $a_0 \neq 0$, as defined by D.H. Lehmer [8] in 1933, is

$$\mathrm{M}(P) = |a_0| \prod_{j=1}^{n} \max(1, |\alpha_j|).$$

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In 1962, K. Mahler [9] gave the following definition

$$\mathrm{M}(P) = \exp\left(\int_0^1 \log|P(e^{2\pi it})|dt\right),$$

which is equivalent to Lehmer's definition by Jensen's formula [6]

$$\int_0^1 \log|e^{2\pi it} - \alpha|dt = \log\max(1, |\alpha|).$$

If α is an algebraic integer, then the Mahler measure of α, denoted by $\mathrm{M}(\alpha)$, is the Mahler measure of its minimal polynomial P in $\mathbb{Z}[z]$. The *absolute Mahler measure of* α is defined by $\Omega(\alpha) = \mathrm{M}(\alpha)^{1/\deg(\alpha)}$.

If α is an algebraic integer and $\mathrm{M}(\alpha) = 1$, then a classical theorem of Kronecker [7] tells us that α is a root of unity. It suggests the question: $\inf_{\alpha \text{ not a root of unity}} \mathrm{M}(\alpha) > 1$? It is known as *Lehmer's problem* and it is still open. Another formulation can be given as follows. Does there exist an absolute constant $c > 0$ such that: if $\mathrm{M}(\alpha) > 1$ then $\mathrm{M}(\alpha) > 1 + c$? The smallest known value is due to Lehmer himself and is $\mathrm{M}(P) = 1.176280\ldots$ where $P(z) = z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$.

In this paper, we are interested in totally positive algebraic integers α, i.e., algebraic integers all of whose conjugates are positive real numbers. Let $\mathcal{L}$ be the set of the quantities $\Omega(\alpha)$ where α is a totally positive algebraic integer. In 1973, A. Schinzel [12] showed that all totally positive algebraic integers α, different from 0 and 1, satisfy $\Omega(\alpha) \geq \dfrac{1+\sqrt{5}}{2}$ and the equality holds if α is a root of the polynomial x^2-3x+1. It means that $\dfrac{1+\sqrt{5}}{2}$ is the smallest element of the spectrum of the absolute Mahler measure of totally positive algebraic integers. In 1981, C. Smyth [13] showed that, if α is a totally positive algebraic integer, then, with a finite set of exceptions, $\Omega(\alpha) \geq 1.717177\ldots$. His result uses the method of explicit auxiliary functions with heuristic search of polynomials and allows one to find the three following points of $\mathcal{L}$. He also showed that $\mathcal{L}$ is dense in $[1.727305\ldots, \infty)$. In 1994, with the same method and thanks to numerical improvements, we obtained [3] that, if α is a positive algebraic integer, then, with a finite set of exceptions, $\Omega(\alpha) \geq 1.720678\ldots$. This lower bound gives the two following points of the spectrum. Our recursive algorithm, developed in [4] from Wu's algorithm [15] substitutes the heuristic search with a systematic search by induction of suitable polynomials used in the auxiliary functions. We prove the following

Theorem 1. *If α is a nonzero totally positive algebraic integer whose minimal polynomial is different from $x-1$, x^2-3x+1, x^2-4x+1, $x^4-7x^3+13x^2-7x+1$, $x^6-11x^5+41x^4-63x^3+41x^2-11x+1$, $x^8-15x^7+83x^6-220x^5+303x^4-220x^3+83x^2-15x+1$,*