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Prime polynomial values of linear functions in short intervals

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ABSTRACT

In this paper we establish a function field analogue of a conjecture in number theory which is a combination of several famous conjectures, including the Hardy–Littlewood prime tuple conjecture, conjectures on the number of primes in arithmetic progressions and in short intervals, and the Goldbach conjecture. We prove an asymptotic formula for the number of simultaneous prime polynomial values of n linear functions, in the limit of a large finite field.

1. Introduction

Recently, several function field analogues of problems in analytic number theory were solved in the limit of a large finite field, e.g. the Bateman–Horn conjecture [9]; the Goldbach conjecture [5]; the Chowla conjecture [8]; problems on variance of the number of primes in short intervals and in arithmetic progressions [11] and on covariance of almost primes [13].

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Let us describe in more detail two classical problems in number theory and their resolutions in the function field case. The two problems we describe relate to the work of this paper. Let $\mathbb{1}$ be the prime characteristic function, i.e.,

$$\mathbb{1}(h) = \begin{cases} 1, & h \text{ is prime} \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

The first problem is of counting primes in short intervals. By the Prime Number Theorem, it is conjectured that if I is an interval of length x^ϵ, $\epsilon > 0$, around large number x, then

$$\sum_{h\in I} \mathbb{1}(h) \sim \int_I \frac{dt}{\log t} \sim \frac{x^\epsilon}{\log x}. \tag{2}$$

Let $\mathbb{F}_q[t]$ be the ring of polynomials over the finite field $\mathbb{F}_q$ with q elements. By abuse of notation, denote by $\mathbb{1}$ the analogue of (1), i.e., the characteristic function of prime polynomials (which are by definition monic irreducibles), and let $\|f\| = q^{\deg f}$, for $f \in \mathbb{F}_q[t]$ (where $\|0\| = 0$). Rosenzweig and the authors [2] prove the following analogue of (2): Let $f_0 \in \mathbb{F}_q[t]$ be monic of degree k, $\frac{3}{k} \le \epsilon < 1$, and $I = I(f_0, \epsilon) = \{f \in \mathbb{F}_q[t] : \|f - f_0\| \le \|f_0\|^\epsilon\}$; then

$$\sum_{f\in I} \mathbb{1}(f) = \frac{\#I}{k}(1 + O_k(q^{-1/2})), \tag{3}$$

where the implied constant depends only on k and not on f_0 or q. To compare between (2) and (3), we replace x^ϵ with $\#I$, and $\log x$ with k.

The second problem is the Hardy–Littlewood prime tuple conjecture, which asserts that

$$\sum_{0<h\le x} \mathbb{1}(h + a_1)\cdots\mathbb{1}(h + a_n) \sim \mathfrak{S}(a_1, \dots, a_n)\frac{x}{(\log x)^n}, \qquad x \to \infty, \tag{4}$$

where

$$\mathfrak{S}(a_1, \dots, a_n) = \prod_p \frac{1 - \nu(p)p^{-1}}{(1 - p^{-1})^n},$$

and $\nu(p) = \#\{h \bmod p : (h + a_1)\cdots(h + a_n) \equiv 0 \pmod p\}$. Note that $\mathfrak{S} = 0$ if and only if $\nu(p) = p$ for some p, which implies that p divides $(h + a_1)\cdots(h + a_n)$ for all h. Bender and Pollack [6] in the case $n = 2$ and the second author [4] in general, prove that for any fixed $k > 0$

$$\sum_{\substack{f\in\mathbb{F}_q[t] \text{ monic} \\ \deg f = k}} \mathbb{1}(f + a_1)\cdots\mathbb{1}(f + a_n) = \frac{q^k}{k^n}(1 + O_{k,n}(q^{-1/2})),$$