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Centers and characters of Jacobi group-invariant differential operator algebras

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ABSTRACT

We study the algebras of differential operators invariant with respect to the scalar slash actions of real Jacobi groups of arbitrary rank. We consider only slash actions with invertible indices. The corresponding algebras are non-commutative and are generated by their elements of orders 2 and 3. We prove that their centers are polynomial in one variable and are generated by the Casimir operator. We also compute their characters: in rank exceeding 1 there are two, and in rank 1 there are in general five. In rank 1 we compute in addition all of their irreducible admissible representations.

1. Introduction

The real Jacobi group G^J_N of degree 1 and rank N is the semidirect product of $\mathrm{SL}_2(\mathbb{R})$ acting on a certain Heisenberg central extension of N copies of the standard representation. The theory of Jacobi forms on G^J_1 was developed in [EZ85] and has since been widely

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used. The generalization to Jacobi forms of higher degree and rank, and in particular to Jacobi forms on G_N^J, was initiated in [Zi89].

Jacobi forms are holomorphic by definition. In the definition of automorphic forms on general semisimple Lie groups, holomorphicity is replaced by the condition that the form be an eigenfunction of all elements of the center of the universal enveloping algebra of the group [HC59], or at least that it generate a finite dimensional representation of the center [Bo66]. Berndt and Schmidt [BS98] proposed a similar approach for the Jacobi group, which of course is not semisimple: they defined automorphic forms on G_1^J using the cubic invariant operator C defined in [BB90] in place of the center. This definition was used in the study of non-holomorphic Maaß–Jacobi forms in [Pi09] and [BR10]. At this time it was realized that the center of the universal enveloping algebra of G_1^J in fact acts by $\mathbb{C}[C]$ [BCR12]. Thus the definition from [BS98] coincides with the classical one.

In order to describe the representation-theoretic framework of automorphic forms, let G be a connected real Lie group, K a Lie subgroup, and V a complex representation of K. In this setting we have the G-vector bundle $G \times_K V$ over the homogeneous space G/K, and the space $C^\infty_{\mathrm{sec}}(G \times_K V)$ of smooth sections of $G \times_K V$. The group G acts on $C^\infty_{\mathrm{sec}}(G \times_K V)$ by translations, its complexified Lie algebra $\mathfrak{g}$ acts by vector fields, and its universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ acts by differential operators. In the event that the vector bundle is topologically trivial, its sections may be identified with the V-valued functions. In this case the associated right action of G on the functions is called a *slash action*.

Observe that the center $\mathfrak{Z}(\mathfrak{g})$ of $\mathfrak{U}(\mathfrak{g})$ acts on $C^\infty_{\mathrm{sec}}(G \times_K V)$ by differential operators which commute with the action of G. In general, such differential operators are said to be *invariant*. The *invariant differential operator algebra* of $G \times_K V$ (its "IDO algebra") is the associative algebra of all invariant differential operators on $C^\infty_{\mathrm{sec}}(G \times_K V)$. We will denote it by

$$\mathbb{D}(G \times_K V).$$

This algebra is not necessarily commutative, and we write $Z(\mathbb{D}(G \times_K V))$ for its center. It is a crucial point that the $\mathfrak{Z}(\mathfrak{g})$-action manifestly commutes with $\mathbb{D}(G \times_K V)$ as well as with the G-action, so there is a natural homomorphism

$$\mathfrak{Z}(\mathfrak{g}) \to Z\big(\mathbb{D}(G \times_K V)\big).$$

In general, this homomorphism is neither injective nor surjective.

IDO algebras have been the subject of many deep investigations; see for example the survey articles [He77] and [He79] and the references therein. One focus is on *eigenspace representations*: if $\chi : \mathbb{D}(G \times_K V) \to \mathbb{C}$ is a *character*, *i.e.*, a homomorphism onto the scalars, the associated eigenspace representation of G is

$$C^\infty_{\mathrm{sec}}(G \times_K V)_\chi := \big\{ s \in C^\infty_{\mathrm{sec}}(G \times_K V) : Ds = \chi(D)s \ \forall D \in \mathbb{D}(G \times_K V) \big\}.$$