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Exponential sums over primes in short intervals

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A R T I C L E I N F O

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A B S T R A C T

Let $\Lambda(n)$ be the von Mangoldt function, x real and $2 \le y \le x$. This paper improves the estimate on the exponential sum over primes in short intervals

$$S_k(x, y; \alpha) = \sum_{x<n\le x+y} \Lambda(n)e\big(n^k\alpha\big)$$

when $k \ge 4$ for all $\alpha \in [0, 1]$. And then combined with the Hardy–Littlewood method, this enables us to give some short interval variants of Hua's theorems in additive prime number theory.

1. Introduction and statement of results

Let $\Lambda(n)$ be the von Mangoldt function, $k \ge 1$ an integer, x real and $2 \le y \le x$. The estimate of the exponential sum over primes in short intervals

$$S_k(x, y; \alpha) = \sum_{x<n\le x+y} \Lambda(n)e\big(n^k\alpha\big) \tag{1.1}$$

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was first studied by I.M. Vinogradov [11] in 1939 with his elementary method. Since then this topic has attracted the interest of quite a number of authors (see [1,5–10,12], etc.). These sums arise naturally and play important roles when solving the Waring–Goldbach problems in short intervals by the circle method. In particular, the case $k = 1$, i.e., the linear exponential sum over primes in short intervals, was studied quite extensively, because of its applications to the study of the Goldbach–Vinogradov theorem with three almost equal prime variables (see [12] and the references therein).

For the case $k = 2$, Liu and Zhan [7] first established a non-trivial estimate of $S_2(x, y; \alpha)$ for all α and all published results before their result are valid only for α in a very thin subset of $[0, 1]$. In [8], Lü and Lao improved the results in [7] to be as good as what was previously derived from the Generalized Riemann Hypothesis.

In this paper we deal with $S_k(x, y; \alpha)$ for all $\alpha \in [0, 1]$ in the general case $k \geq 3$. In [6], Liu and Zhan first established a non-trivial estimate of $S_k(x, y; \alpha)$ for all $\alpha \in \mathbb{R}$ and $k \geq 3$. To state Liu and Zhan's result, we introduce some notation. Let $A > 0$ be any given large constant, $\varepsilon > 0$ sufficiently small. We further put

$$L = \log x, \qquad P = L^{c_1}, \qquad \mathcal{P} = x^{k\varrho}, \qquad Q = \frac{y^{2k-1}}{x^{k-1}} L^{-c_3}, \qquad R = yx^{k-1}L^{c_2}, \tag{1.2}$$

such that

$$2 \leq 2P < 2\mathcal{P} < Q \leq R \leq x^k, \tag{1.3}$$

where ϱ is a positive parameter depending on k which will be specified later and c_i denote positive constants that depend at most on the positive numbers A, k and ε. By Dirichlet's lemma on rational approximation, any $\alpha \in [0, 1]$ can be written as

$$\alpha = \frac{a}{q} + \lambda, \quad \text{with} \quad (a, q) = 1, \quad 1 \leq a \leq q \leq Q, \qquad |\lambda| \leq \frac{1}{qQ}. \tag{1.4}$$

Then every $\alpha \in [0, 1]$ given in the form of (1.4) satisfies one of the following three conditions:

$$\begin{aligned}
&(a) \quad q \leq P, && |\lambda| \leq \frac{1}{R};\\
&(b) \quad P < q \leq Q, && |\lambda| \leq \frac{1}{qQ};\\
&(c) \quad q \leq P, && \frac{1}{R} < |\lambda| \leq \frac{1}{qQ}.
\end{aligned}$$

Denote by $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ the three subsets of α satisfying (a), (b) and (c) respectively. Then $[0, 1]$ is the disjoint union of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. The main result in [6] is the following