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Subsequence sums: Direct and inverse problems

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ABSTRACT

Let $A = (\underbrace{a_0, \ldots, a_0}_{r_0 \text{ copies}}, \underbrace{a_1, \ldots, a_1}_{r_1 \text{ copies}}, \ldots, \underbrace{a_{k-1}, \ldots, a_{k-1}}_{r_{k-1} \text{ copies}})$ be a finite sequence of integers with k distinct terms, denoted alternatively by $(a_0, a_1, \ldots, a_{k-1})_{\bar{\mathbf{r}}}$, where $a_0 < a_1 < \cdots < a_{k-1}$, $\bar{\mathbf{r}} = (r_0, r_1, \ldots, r_{k-1})$, $r_i \geq 1$ for $i = 0, 1, \ldots, k-1$. The sum of all the terms of a subsequence of length at least one of the sequence A is said to be a subsequence sum of A. The set of all subsequence sums of A is denoted by $S(\bar{\mathbf{r}}, A)$. The direct problem for subsequence sums is to find the lower bound for $|S(\bar{\mathbf{r}}, A)|$ in terms of the number of distinct terms in the sequence A. The inverse problem for subsequence sums is to determine the structure of the finite sequence A of integers for which $|S(\bar{\mathbf{r}}, A)|$ is minimal. In this paper, both the problems are solved and some well-known results for subset sum problem are obtained as corollaries of the results for subsequence sum problem.

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0. Notation and terminology

For integers a, b, c and for a set S of integers, let $[a,b] = \{n \in \mathbb{Z} : a \leq n \leq b\}$, $c * S = \{ca : a \in S\}$ and $c + S = \{c + a : a \in S\}$. Let $A = (\underbrace{a_0,\dots,a_0}_{r_0 \text{ copies}}, \underbrace{a_1,\dots,a_1}_{r_1 \text{ copies}}, \dots, \underbrace{a_{k-1},\dots,a_{k-1}}_{r_{k-1} \text{ copies}})$ be a finite sequence of integers, where $a_0 < a_1 < \cdots < a_{k-1}$ and $r_i \geq 1$ for $i = 0, 1, \dots, k-1$. Let $\bar{\mathbf{r}} = (r_0, r_1, \dots, r_{k-1})$ be the ordered k-tuple. Then we denote the sequence $A = (\underbrace{a_0,\dots,a_0}_{r_0 \text{ copies}}, \underbrace{a_1,\dots,a_1}_{r_1 \text{ copies}}, \dots, \underbrace{a_{k-1},\dots,a_{k-1}}_{r_{k-1} \text{ copies}})$ by $(a_0, a_1, \dots, a_{k-1})_{\bar{\mathbf{r}}}$. In this paper, by a finite sequence of integers, we always mean a finite sequence of integers whose terms are in nondecreasing order. This does not cause any harm, because the results in this paper are related to subsequence sums which remain unaltered by changing the order of the terms of the sequence. For an integer c, let $c + (a_0, a_1, \dots, a_{k-1})_{\bar{\mathbf{r}}} = (c + a_0, c + a_1, \dots, c + a_{k-1})_{\bar{\mathbf{r}}}$ and for a nonzero integer c, let

$$c * A = c * (a_0, a_1, \dots, a_{k-1})_{\bar{\mathbf{r}}} = \begin{cases} (ca_0, ca_1, \dots, ca_{k-1})_{\bar{\mathbf{r}}}, & \text{if } c > 0, \\ (ca_{k-1}, ca_{k-2}, \dots, ca_0)_{\bar{\mathbf{s}}}, & \text{if } c < 0, \end{cases}$$

where $\bar{\mathbf{s}} = (r_{k-1}, \dots, r_0)$. For integers a, b and c such that $b = a + k - 1$, let

$$[a,b]_{\bar{\mathbf{r}}} = (a, a+1, \dots, b)_{\bar{\mathbf{r}}},$$

and

$$c + [a,b]_{\bar{\mathbf{r}}} = [c+a, c+b]_{\bar{\mathbf{r}}}.$$

We shall call $[a,b]_{\bar{\mathbf{r}}}$ a *sequence-interval*. In particular, if $\bar{\mathbf{r}} = (r, r, \dots, r)$, then we simply write $(a_0, a_1, \dots, a_{k-1})_r$ in place of $(a_0, a_1, \dots, a_{k-1})_{\bar{\mathbf{r}}}$ and $[a,b]_r$ in place of $[a,b]_{\bar{\mathbf{r}}}$. Let

$$[a,b]_r \backslash (0)_r = (a, a+1, \dots, -1, 1, 2, \dots, b)_r.$$

If A is a sequence, then we write $a \in A$ to mean that a is a term of the sequence A. If A is a set, then the notation $a \in A$ has the usual meaning. We say that two finite sequences of integers are *disjoint* if there is no common term in these sequences. In this case, we write $A \cup B$ to denote the sequence obtained by merging these sequences and reordering the terms in nondecreasing order. For example, if $A = (1,2,4)_{\bar{\mathbf{u}}}$, where $\bar{\mathbf{u}} = (2,1,3)$ and $B = (3,5,6)_{\bar{\mathbf{v}}}$, where $\bar{\mathbf{v}} = (1,2,2)$ are two sequences, then $A \cup B = (1,1,2,4,4,4) \cup (3,5,5,6,6) = (1,2,3,4,5,6)_{\bar{\mathbf{r}}}$, where $\bar{\mathbf{r}} = (2,1,1,3,2,2)$. If A and B are two sets, then $A \cup B$ has usual meaning. Finally, we agree with the convention that $\binom{a}{b} = 0$ if a and b are two positive integers such that $a < b$.