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Distribution of Artin–Schreier–Witt extensions

Thorsten Lagemann

ERGO Versicherungsgruppe, Aachener Straße 300, 50933 Köln, Germany

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A B S T R A C T

The article at hand contains exact asymptotic formulas for the distribution of conductors of abelian p-extensions of global function fields of characteristic p. These yield a new conjecture for the distribution of discriminants fueled by an interesting lower bound.

1. Main results

Let F be a global function field of characteristic p, that is a transcendental extension of degree 1 over some finite field $\mathbb{F}_q$ of cardinality q, and let $\mathbb{F}_q$ be algebraically closed in F. For an extension E/F, let $\mathfrak{f}(E/F)$ be its relative conductor, and $\|\mathfrak{f}(E/F)\| = q^{\deg \mathfrak{f}(E/F)}$ its absolute norm. The object of interest is the counting function

$$C(F, G; X) = \left|\left\{E/F : \mathrm{Gal}(E/F) \simeq G,\ \left\|\mathfrak{f}(E/F)\right\| \leq X\right\}\right|$$

of field extensions E/F in a fixed algebraic closure of F with given finite Galois group G and bounded conductor. The values of this function coincide with finite coefficient sums

E-mail address: thorsten.lagemann@gmx.de.

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of the Dirichlet series

$$\Phi(F,G;s) = \sum_{\mathrm{Gal}(E/F)\simeq G} \left\|\mathfrak{f}(E/F)\right\|^{-s},$$

generated by the extensions E/F with Galois group G. For the article at hand, we are interested in field extensions with abelian Galois group of exponent p^e. These extensions are called Artin–Schreier–Witt extensions.

Theorem 1.1. *Let F be a global function field of characteristic p and G be the finite abelian p-group with exponent p^e and p^i-ranks $r_i(G) = \log_p(p^{i-1}G : p^iG)$. Let*

$$\alpha_p(G) = \frac{1 + (p-1)\sum_{i=1}^{e} p^{e-i} r_i(G)}{p^e}$$

and

$$\beta(F,G) = \begin{cases} p^e - 1, & \text{if } G \text{ is cyclic},\\ p^{e-f}, & \text{elsewise, where } 1 \le f \le e \text{ is minimal such that } p^f G \text{ is cyclic.} \end{cases}$$

Then the Dirichlet series $\Phi(F,G;s)$ has convergence abscissa $\alpha_p(G)$, and it possesses a meromorphic continuation, such that $\Phi(F,G;s)\prod_{l=2}^{p^e}\zeta_F(ls - \sum_{i=1}^{e} w_i(l)r_i(G))$ is holomorphic for $\Re(s) > \alpha_p(G) - 1/(2p^e)$, where $\zeta_F(s)$ denotes the zeta function of F and $w_i(l) = \lfloor (l-1)/p^{i-1}\rfloor - \lfloor (l-1)/p^i\rfloor$. The periodic pole $s = \alpha_p(G)$ with period $2\pi i/\log(q)$ has order $\beta(F,G)$, whereas the order of every other pole on the axis $\Re(s) = \alpha_p(G)$ does not exceed $\beta(F,G)$.

The series $\Phi(F,G;s)$ is a power series in $t = q^{-s}$, which is convergent for $|t| < q^{-\alpha_p(G)}$, and its poles are located at $t = \xi q^{-\alpha_p(G)}$ for finitely many roots of unity ξ. If c_n denotes the n-th power series coefficient of $\Phi(F,G;s)$, we obtain $C(F,G;q^m) = \sum_{n=0}^{m} c_n$. By an application of the Cauchy integral formula, which is elaborated in Theorem A.5 in [7], this coefficient sum has the following asymptotic behaviour.

Theorem 1.2. *Let F be a global function field of characteristic p and G be a finite abelian p-group. Then the number of Artin–Schreier–Witt extensions E/F with Galois group G has asymptotic*[1]

$$C(F,G;X) \sim \gamma(F,G) X^{\alpha_p(G)} \log(X)^{\beta(F,G)-1}$$

with some explicitly computable constant $\gamma(F,G) > 0$.

[1] Hereby the asymptotic equivalence $f(X) \sim g(X)$ means that there are integers $\ell \ge 1$ and $e \ge 0$ such that $\lim_{n\to\infty} f(q^{\ell n+e})/g(q^{\ell n+e}) = 1$ holds. The main reason for introducing this slightly weaker definition is, that the counting functions $f(X)$ in question are step functions in $X = q^n$, where n runs over some arithmetic progression. Hence, the asymptotic quotient of $f(X)/g(X)$ would not exist for continuous functions $g(X)$. In order to compare function field asymptotics with number field asymptotics, we use this redefinition regarding the other Landau symbols as well. See [7] for comments and examples.