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On distinct unit generated fields that are totally complex

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We consider the problem of characterizing all number fields K such that all algebraic integers $\alpha \in K$ can be written as the sum of distinct units of K. We extend a method due to Thuswaldner and Ziegler [12] that previously did not work for totally complex fields and apply our results to the case of totally complex quartic number fields.

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1. Introduction

Jacobson [9] observed in the 1960's that the two number fields $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{5})$ share the property that every algebraic integer is the sum of distinct units. Moreover, he conjectured that these two quadratic number fields are the only quadratic number fields with this property. Let us call a field with this property a distinct unit generated field or DUG-field for short.

In the 1970's Śliwa [11] solved this problem for quadratic number fields and showed that even no pure cubic number field is DUG. These results have been extended to cubic and quartic fields by Belcher [3,4]. In particular, Belcher solved the case of imaginary cubic number fields completely [4].

The problem of characterizing all number fields in which every algebraic integer is a sum of distinct units is still unsolved. Let us note that this problem is contained in Narkiewicz' list of open problems in his famous book [10, see page 539, Problem 18].

Recently Thuswaldner and Ziegler [12] used methods originating from the theory of number systems and enumeration and obtained a new approach to the problem and introduced the following definition in order to measure how far is a number field away from being a DUG-field.

Definition 1. Let $\mathfrak{o}$ be some order in a number field K and $\alpha \in \mathfrak{o}$. Suppose α can be written as a linear combination of units

$$\alpha = a_1\epsilon_1 + \cdots + a_\ell\epsilon_\ell,$$

such that $\epsilon_1, \ldots, \epsilon_\ell \in \mathfrak{o}^*$ are all distinct and $a_1 \geq \cdots \geq a_\ell > 0$ are positive integers. Choose a representation with a_1 minimal, then we call $\omega(\alpha) = a_1$ the unit sum height of α. Moreover we define $\omega(0) = 0$ and $\omega(\alpha) = \infty$ if α is not the sum of units.

We define

$$\omega(\mathfrak{o}) = \max\{\omega(\alpha) \,:\, \alpha \in \mathfrak{o}\}$$

if the maximum exists. If it does not exist we write $\omega(\mathfrak{o}) = \omega$ in case of $\mathfrak{o}$ is generated by units and $\omega(\mathfrak{o}) = \infty$ otherwise.

In case of $\mathfrak{o}$ is the maximal order of K we also write $\omega(K) = \omega(\mathfrak{o})$.

Unfortunately the method of Thuswaldner and Ziegler [12] only works for number fields which have a real embedding, i.e. which are not totally complex. Such fields contain a Pisot unit, which is essential for the tool provided there. Recall that an algebraic integer $\alpha > 1$ is a Pisot number, if all its conjugates are of modulus less than 1.

On the other hand, Hajdu and Ziegler [8] focused on totally complex fields. For the case of quartic totally complex fields they provided the list of candidates of DUG fields (see Table 1), where ζ_μ denotes a primitive μ-th root of unity.