Journal of Number Theory 148 (2015) 328–364

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

The elliptic dilogarithm for the sunset graph [☆]

Spencer Bloch [a], Pierre Vanhove [b,c,d,*]

[a] *5765 S. Blackstone Ave., Chicago, IL 60637, USA*
[b] *Institut des Hautes Etudes Scientifiques, Le Bois-Marie, 35 route de Chartres, F-91440 Bures-sur-Yvette, France*
[c] *Institut de Physique Théorique, CEA, IPhT, F-91191 Gif-sur-Yvette, France*
[d] *CNRS, URA 2306, F-91191 Gif-sur-Yvette, France*

A R T I C L E   I N F O

Article history:
Received 15 January 2014
Received in revised form 19 September 2014
Accepted 19 September 2014
Available online 4 November 2014
Communicated by David Goss

Keywords:
Feynman graph
Variation of mixed hodge structures
Motives
Elliptic dilogarithm
Eisenstein series

A B S T R A C T

We study the sunset graph defined as the scalar two-point self-energy at two-loop order. We evaluated the sunset integral for all identical internal masses in two dimensions. We give two calculations for the sunset amplitude; one based on an interpretation of the amplitude as an inhomogeneous solution of a classical Picard–Fuchs differential equation, and the other using arithmetic algebraic geometry, motivic cohomology, and Eisenstein series. Both methods use the rather special fact that the amplitude in this case is a family of periods associated to the universal family of elliptic curves over the modular curve $X_1(6)$. We show that the integral is given by an elliptic dilogarithm evaluated at a sixth root of unity modulo periods. We explain as well how this elliptic dilogarithm value is related to the regulator of a class in the motivic cohomology of the universal elliptic family.

Contents

[☆] IPHT-T/13/217, IHES/P/13/24.

* Corresponding author.
E-mail addresses: spencer_bloch@yahoo.com (S. Bloch), pierre.vanhove@cea.fr (P. Vanhove).

http://dx.doi.org/10.1016/j.jnt.2014.09.032
0022-314X/

1. Introduction

Scattering amplitudes are fundamental objects describing how particles interact. At a given loop order in the perturbative expansion in the coupling constant, there are many ways of constructing the amplitudes from first principles of quantum field theory. The result is an algebraic integral with parameters, and the physical problem of efficient evaluation of the integral is linked to the qualitative mathematical problem of classifying these multi-valued functions of the complexified kinematic invariants. The amplitudes are locally analytic, presenting branch points at the thresholds where particles can appear.

These questions can be studied order by order in perturbation. At one-loop order, around a four dimensional space–time, all the scattering amplitudes can be expanded in a basis of integral functions given by box, triangle, bubbles and tadpole integral functions, together with rational term contributions [7,13,41] (see [6,12] for some reviews on this subject).

The finite part of the $\epsilon = (4 - D)/2$ expansion of the box and triangle integral functions is given by dilogarithms of the proper combination of kinematic invariants. The finite part of the bubble and tadpole integral is a logarithm function of the physical parameters.

The appearance of the dilogarithm and logarithms at one-loop order is predictable from unitarity considerations since this reproduces the behavior of the one-loop scattering amplitude under single, or double two-particle cuts in four dimensions.

The fact that one-loop amplitudes are expressed as dilogarithms and logarithms can as well be understood motivitically [10,11], but the status of two-loop order scattering amplitude is far less well understood for generic amplitudes (see for instance [18,28,29,35] for some recent progress).

The sunset integral arises as the two-loop self-energy diagram in the evaluation of higher-order correction in QED, QCD or electroweak theory precision calculations [4], or as a sub-topology of higher-order computation [18]. As a consequence, it has been the subject of numerous analyses. The integral for various configurations of vanishing masses has been analyzed using the Mellin–Barnes methods in [43], with two different masses and three equal masses in [33]. An asymptotic expansion of the sunset integral has been given in [16]. Various forms for the integral have been considered either in geometrical