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Congruences for the Fourier coefficients of the Mathieu mock theta function

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ABSTRACT

In this paper, we study the congruences for the Fourier coefficients of the Mathieu mock theta function, which appears in the Mathieu moonshine phenomenon discovered by Eguchi, Ooguri, and Tachikawa.

1. Introduction

Let $q = e^{2\pi i\tau}$ for τ in the complex upper half plane. Then the classical modular function $j : \mathbb{H} \to \mathbb{C}$ possesses the following Fourier expansion

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$$j(\tau) = \sum_{n=-1}^{\infty} c(n)q^n = \frac{1}{q} + 744 + 196\,884q + \cdots$$

The moonshine phenomenon discovered by McKay and precisely formulated by Conway and Norton [2] asserts that the Fourier coefficients of the j-function are related to the dimensions of the irreducible representations of the Monster group. Subsequently, the moonshine conjecture was proven by Borcherds [1]. It is remarkable that the coefficients $c(n)$ have the following congruences [5,6] for $n \geq 1$, $a \geq 1$, and $1 \leq b \leq 3$:

$$n \equiv 0 \begin{cases} \pmod{2^a} \\ \pmod{3^a} \\ \pmod{5^a} \\ \pmod{7^a} \\ \pmod{11^b} \end{cases} \quad \Rightarrow \quad c(n) \equiv 0 \begin{cases} \pmod{2^{3a+8}} \\ \pmod{3^{2a+3}} \\ \pmod{5^{a+1}} \\ \pmod{7^a} \\ \pmod{11^b}. \end{cases}$$

In 2010, Eguchi, Ooguri, and Tachikawa [3] discovered a similar phenomenon corresponding to the Mathieu group M_{24}. To describe their observation, we introduce the following functions:

$$\theta_1(z;\tau) = -\sum_{n=-\infty}^{\infty} e^{\pi i\tau(n+\frac{1}{2})^2 + 2\pi i(n+\frac{1}{2})(z+\frac{1}{2})},$$

$$\mu(z;\tau) = \frac{ie^{\pi iz}}{\theta_1(z;\tau)} \sum_{n\in\mathbb{Z}} (-1)^n \frac{q^{\frac{1}{2}n(n+1)} e^{2\pi inz}}{1 - q^n e^{2\pi iz}},$$

and we set $\Sigma(\tau)$ and $A(n)$ as follows:

$$\begin{aligned} \Sigma(\tau) &:= -8 \sum_{z\in\{1/2,\tau/2,(1+\tau)/2\}} \mu(z;\tau) \\ &= -q^{-\frac{1}{8}}\left(2 - \sum_{n=1}^{\infty} A(n)q^n\right) \\ &= q^{-\frac{1}{8}}\left(-2 + 90q + 462q^2 + 1540q^3 + 4554q^4 + 11\,592q^5 + 27\,830q^6 + \cdots\right). \end{aligned}$$

Then the Mathieu moonshine phenomenon is the statement that the first five coefficients appearing in the Fourier expansion divided by 2,

$$\{45, 231, 770, 2277, 5796\},$$

are equal to the dimensions of the irreducible representations of M_{24}, and the other coefficients can be written as linear combinations of the dimensions of the Mathieu group M_{24}, for example $13\,915 = 3520 + 10\,395$. The reason for this mysterious phenomenon is still unknown. In this case the dimensions of the irreducible representations no longer appear in the Fourier coefficients of a modular form but rather in a mock modular form.