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Single-valued multiple polylogarithms and a proof of the zig–zag conjecture

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ABSTRACT

A long-standing conjecture in quantum field theory due to Broadhurst and Kreimer states that the periods of the zig–zag graphs are a certain explicit rational multiple of the odd values of the Riemann zeta function. In this paper we prove this conjecture by constructing a certain family of single-valued multiple polylogarithms which correspond to multiple zeta values $\zeta(2,\ldots,2,3,2,\ldots 2)$ and using the method of graphical functions. The zig–zag graphs are the only infinite family of primitive graphs in ϕ_4^4 theory (in fact, in any renormalisable quantum field theory in four dimensions) whose periods are now known.

1. Introduction

In 1995 Broadhurst and Kreimer [7] conjectured a formula for the Feynman periods of a well-known family of graphs called the zig–zag graphs. We give a proof of this conjecture using the second author's theory of graphical functions [29,27] (see also [17]) and a variant of the first author's theory of single-valued multiple polylogarithms [8].

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The proof makes use of a recent theorem due to Zagier [35,25] on the evaluation of the multiple zeta values $\zeta(2,\dots,2,3,2,\dots,2)$ in terms of the numbers $\zeta(2m+1)\pi^{2k}$.

1.1. A family of single-valued polylogarithms

The classical polylogarithm

$$\mathrm{Li}_n(z) = \sum_{k\geq 1} \frac{z^k}{k^n}, \tag{1.1}$$

is defined for $n \geq 1$ and $|z| < 1$. It extends analytically to a multivalued function on $\mathbb{C}\backslash\{0,1\}$ and admits a canonical single-valued version $\mathcal{L}_n(z)$ [8], where

$$\mathcal{L}_1(z) = 2\log|1-z| \quad \text{and} \quad \mathcal{L}_2(z) = -2i\,\mathrm{Im}\big(\mathrm{Li}_2(z)\big) + 2\log|z|\log(1-\bar{z}).$$

The second function is a close relative of the Bloch–Wigner dilogarithm (1.6). Various different single-valued versions of (1.1) were first constructed in [32,34] and arose in physics in [16]. An important conjecture in arithmetic algebraic geometry due to Zagier [18] states that the regulator on the algebraic K-theory of number fields can be expressed in terms of the single-valued classical polylogarithms. As a result, the mixed Hodge theory of the classical polylogarithms (1.1) has generated an extensive literature, culminating in a motivic interpretation of Zagier's conjecture due to Beilinson and Deligne [2]. The expected interpretation as a regulator requires that the values at $z = 1$ of the functions (1.1) (single Riemann zeta values) correspond to primitive elements in the Lie coalgebra of motivic multiple zeta values, which is indeed the case.

In [8], one of us showed that all multiple polylogarithms in one variable (or equivalently, iterated integrals on the projective line minus 3 points $\mathbb{P}^1\backslash\{0,1,\infty\}$) admit canonical single-valued versions by combining their real and imaginary parts. These functions are denoted by $\mathcal{L}_w(z)$, where w is any word in the alphabet in two letters $\{\mathsf{x}_0,\mathsf{x}_1\}$. The case of the classical polylogarithms (1.1) corresponds to the family of words $w = \mathsf{x}_1\mathsf{x}_0\dots\mathsf{x}_0$. The functions $\mathcal{L}_w(z)$ are uniquely determined by demanding that they be single-valued real-analytic solutions to the Knizhnik–Zamolodchikov equation (in $\frac{\partial}{\partial z}$) with prescribed limiting behaviour at $z = 0$ [8]. It follows from this fact that any differential equation on $\mathbb{P}^1\backslash\{0,1,\infty\}$ with regular singularities and global unipotent monodromy has a single-valued solution in a certain ring generated by the functions $\mathcal{L}_w(z)$. The latter property can be used for evaluating a large class of Feynman integrals [29]. Unfortunately, the general definition of $\mathcal{L}_w(z)$ does not have a closed formula, and, in order to write down families of single-valued polylogarithms one needs a precise knowledge of certain coefficients in the Drinfeld associator.

The majority of this paper is devoted to giving an explicit construction (Section 2) of a somewhat different family of single-valued functions generalising the Bloch–Wigner dilogarithm. The new functions are single-valued solutions to a certain differential equation in both $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$, and their values at 1 are related to the Hoffman multiple zeta