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Mellin transforms with only critical zeros: Legendre functions

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We consider the Mellin transforms of certain Legendre functions based upon the ordinary and associated Legendre polynomials. We show that the transforms have polynomial factors whose zeros lie all on the critical line $\mathrm{Re}\, s = 1/2$. The polynomials with zeros only on the critical line are identified in terms of certain ${}_3F_2(1)$ hypergeometric functions. These polynomials possess the functional equation $p_n(s) = (-1)^{\lfloor n/2 \rfloor} p_n(1-s)$. Other hypergeometric representations are presented, as well as certain Mellin transforms of fractional part and fractional part-integer part functions. The results should be of interest to special function theory, combinatorial geometry, and analytic number theory.

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1. Introduction

Mellin transforms are very important in analytic number theory and asymptotic analysis. They occasionally also find application in signal and image analysis. In a series of investigations, we are determining families of polynomials arising from Mellin transformation that satisfy the Riemann hypothesis.

In particular, we are considering certain Mellin transforms comprised of classical orthogonal polynomials that yield polynomial factors with zeros only on the critical line $\mathrm{Re}\, s = 1/2$ or else only on the real axis. Such polynomials have many important applications to analytic number theory, in a sense extending the Riemann hypothesis. For example, using the Mellin transforms of Hermite functions, Hermite polynomials multiplied by a Gaussian factor, Bump and Ng [8] (see also [7]) were able to generalize Riemann's second proof of the functional equation of the zeta function $\zeta(s)$, and to obtain a new representation for it. The polynomial factors turn out to be certain ${}_2F_1(2)$ Gauss hypergeometric functions, being certain shifted symmetric Meixner–Pollaczek polynomials [11].

The polynomials $\tilde{p}_n(x) = {}_2F_1(-n, -x; 1; 2) = (-1)^n {}_2F_1(-n, x+1; 1; 2)$ and $\tilde{q}_n(x) = i^n n! \tilde{p}_n(-1/2 - ix/2)$ have been studied for combinatorial and number theoretic reasons [17,20], and they directly correspond to the Bump and Ng polynomials with $s = -x$. We note that these polynomials arise in the counting of the number of lattice points in an n-dimensional octahedron [7,20]. In fact, combinatorial, geometrical, and coding aspects of $\tilde{p}_n(x)$ at integer argument had been noted in [15] and [19], and Lemmas 2.2 and 2.3 of [17] correspond very closely to Lemmas 2 and 3, respectively, of [19]. For the half-line Mellin transform of Laguerre functions, one may see [11].

We expect that our work will have connections with the counting of lattice points in polytopes, thus with combinatorial geometry, a polytope being a region described by a set of linear inequalities. In this context, the Ehrhart polynomial [5,18] counts lattice points, and it has a functional equation. Along with the Ehrhart polynomial one may associate a Poincaré series, of the form $P(t) = U(t)/(1-t)^n$, with U a polynomial such that $U(1) \neq 0$. An example form of Poincaré series is

$$P(t) = \frac{\prod_{j=1}^{k}(1 + t + \ldots + t^{n_j})}{(1-t)^n},$$

wherein $n_1, \ldots, n_k$ are positive integers. We expect that other various Ehrhart polynomials are of hypergeometric form, and have all of their zeros on a line.

The Riemann zeta function arises as the half-line Mellin transform of a theta function, and of many other functions. However, these are not the sole type of Mellin transform from which the zeta function may be determined. Letting $\{x\}$ denote the fractional part of x, we quickly review the representation for $\mathrm{Re}\, s > 1$,