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A new generalization of Fermat's Last Theorem ☆

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In this paper, we consider some hybrid Diophantine equations of addition and multiplication. We first improve a result on new Hilbert–Waring problem. Then we consider the equation

$$\begin{cases} A+B=C \\ ABC=D^n \end{cases} \tag{1}$$

where $A, B, C, D, n \in \mathbb{Z}_+$ and $n \geq 3$, which may be regarded as a generalization of Fermat's equation $x^n + y^n = z^n$. When $\gcd(A, B, C) = 1$, (1) is equivalent to Fermat's equation, which means it has no positive integer solutions. We discuss several cases for $\gcd(A, B, C) = p^k$ where p is an odd prime. In particular, for $k = 1$ we prove that (1) has no nonzero integer solutions when $n = 3$ and we conjecture that it is also true for any prime $n > 3$. Finally, we consider Eq. (1) in quadratic fields $\mathbb{Q}(\sqrt{t})$ for $n = 3$.

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1. Introduction

In this paper, we consider some hybrid Diophantine equations of addition and multiplication. First of all,

$$n = x_1 + x_2 + \cdots + x_s$$

such that

$$x_1 x_2 \cdots x_s = x^k,$$

for $n, x_i, x, k \in \mathbb{Z}_+$, which is a new variant of Waring's problem:

$$n = x_1^k + x_2^k + \cdots + x_s^k.$$

We denote by $g'(k)$ (resp. $G'(k)$) the least positive integer such that every integer (resp. all sufficiently large integer) can be represented as a sum of at most $g'(k)$ (resp. $G'(k)$) positive integers, and the product of the $g'(k)$ (resp. $G'(k)$) integers is a k-th power. We show [2] that

$$g'(k) = 2k - 1; \qquad G'(p) \le p + 1;$$
$$G'(2p) \le 2p + 2 \quad (p \ge 3); \qquad G'(4p) \le 4p + 2 \quad (p \ge 7);$$

where k is a positive integer and p is prime. In this paper, we improve the results on composite numbers as follows.

Theorem 1. *For any composite number k, $G'(k) \le k + 2$.*

Next, we consider Fermat's Last Theorem. In 1637, Fermat claimed that the Diophantine equation

$$x^n + y^n = z^n$$

has no positive integer solutions for any integer $n \ge 3$. This was proved finally by Andrew Wiles in 1995 [11,14].

There are several generalizations of Fermat's Last Theorem, e.g., Fermat–Catalan conjecture, which states that the equation $a^m + b^n = c^k$ has only finitely many solutions (a, b, c, m, n, k), where a, b, c are positive coprime integers and m, n, k are positive integers, satisfying $\frac{1}{m} + \frac{1}{n} + \frac{1}{k} < 1$. So far there are only 10 solutions found [4,10]. Meanwhile, Beal's conjecture [8] states that the equation $A^x + B^y = C^z$ has no solution in positive integers A, B, C, x, y and z with x, y and z at least 3 and A, B and C coprime. Beal has offered a prize of one million dollars for a proof of his conjecture or a counterexample [12]. Obviously, there are only finite solutions for Beal's equation under Fermat–Catalan conjecture. Meanwhile, it's known that both FLT and Fermat–Catalan conjecture are the