Journal of Number Theory 149 (2015) 46–56

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

On a generalization of a theorem by Euler

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ARTICLE INFO

Article history:
Received 29 May 2014
Received in revised form 6 October 2014
Accepted 7 October 2014
Available online 27 November 2014
Communicated by David Goss

MSC:
primary 11B68, 42A16, 11B75
secondary 11A67, 42A20, 42A24

Keywords:
Bernoulli numbers
Bernoulli polynomials
Identities
Trigonometric series

ABSTRACT

In this paper a natural generalization of a theorem by Euler in 1744 is presented. Extensive searches failed to locate this result in existing literature or in well known mathematical websites such as MathWorld (http://mathworld.wolfram.com), nor could it be derived by using software for analytical computation like Maple. The obtained identity is fascinating and surprisingly simple, and it paves the way for interesting applications.

1. Introduction

The heritage of the work conducted by Leonhard Euler (1707–1783), the key figure in 18th-century mathematics, and one who should be ranked with Archimedes, Newton and Gauss, seems to be remarkably vivid and alive in modern scientific research areas. For example, in analytic number theory, in algebraic geometry, in topology, in the calculus of variations, and in analysis (both conceptual and numerical), not to mention mechanics

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[1] This research was financially supported by the National Research Foundation of South Africa.

http://dx.doi.org/10.1016/j.jnt.2014.10.002

of particles and solid bodies, astronomy, hydrodynamics, the ideas that Euler generated are still motivating mathematicians, more than three hundred years after his birth.

Kline [8] remarks that "Euler first undertook work on infinite series around 1730, and by that time, John Wallis, Isaac Newton, Gottfried Leibniz, Brook Taylor and Colin Maclaurin had demonstrated the series calculation of the constants e and π and the use of infinite series to represent functions in order to integrate those that could not be treated in closed form. Hence it is understandable that Euler should have tackled the subject". In his paper Kline discusses some interesting examples to illustrate how Euler overcame the difficulties he encountered in his recorded work on infinite series.

In an interesting book by Nahin [9], the author writes on p. 129: "For example, in a 1744 letter to a friend, Euler wrote the following remarkable claim:

$$\frac{\pi - t}{2} = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} = \frac{\sin(t)}{1} + \frac{\sin(2t)}{2} + \frac{\sin(3t)}{3} + \ldots.$$

This was probably (almost certainly) the first 'Fourier series', although of course Euler didn't call it that since Fourier wouldn't be born until twenty-four years later".

In Chapter 4 of his book, Nahin [9] shows how Euler derived the above beautiful formula and also provides the reader with a modern derivation based on Fourier series analysis.

The aim of this paper is to generalize Euler's result, by determining a set of real-valued functions $\{f_k(t), k = 1, 2, \ldots\}$ that has a Fourier series expansion of the form

$$f_k(t) = \sum_{n=1}^{\infty} \frac{\sin^k(nt)}{n^k},$$

for some fixed t.

The elegant identity which is proved in the theorem below is unexpectedly simple in structure and opens the way to derive new results in, for example, algebraic infinite series of numbers and in integrals of functions of Brownian motion and Brownian bridge processes.

The rest of the paper is organized as follows. In Section 2 we provide a brief historical discussion surrounding the present investigation, introduce necessary notation and state some known results in the literature that will be required. Section 3 contains a proof of the main theorem, and Section 4 is devoted to some applications and conclusions.

2. Historical background

The well known Bernoulli numbers $B_0, B_1, B_2, \ldots,$ were mentioned (without using their present names and notation) by Jacob I. Bernoulli (1654–1705) in his posthumous *Ars conjectandi* of 1713 (see, for example, [2]) who was studying the subject of probability at that time. In fact, he derived the following general formula,