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A new look on the generating function for the number of divisors

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ABSTRACT

The q-binomial coefficients are specializations of the elementary symmetric functions. In this paper, we use this fact to give a new expression for the generating function of the number of divisors. As corollaries, we obtained new connections between partitions and divisors.

1. Introduction

Any series of the form

$$\sum_{n=1}^{\infty} a_n \frac{q^n}{1-q^n}, \quad |q| < 1,$$

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where the a_n $(n = 1, 2, \ldots)$ are real numbers is called a Lambert series. These series are well known class of series in analytic function theory and number theory and are mentioned in the classical texts by Abramowitz and Stegun [1, pp. 826–827], Bromwich [7, pp. 102–103], Chrystal [8, pp. 345–346], Hardy and Wright [13, pp. 257–258], Knopp [18, pp. 448–452], MacMahon [24, pp. 26–32], Pólya and Szegő [28, pp. 125–129], and Titchmarsh [30, pp. 160–161]. Lambert series have been elegantly used in a variety of contexts of Ramanujan's research works. The dimension provided by Ramanujan inspired Andrews and Berndt [5] to prove a lot of identities given by Ramanujan.

Lambert series are natural generalizations of the following formula related to the theory of numbers:

$$\sum_{n=1}^{\infty} \frac{q^n}{1-q^n} = \sum_{n=0}^{\infty} \tau(n) q^n, \quad |q| < 1. \tag{1}$$

In multiplicative number theory, the divisor function $\tau(n)$ is defined as the number of divisors of n, unity and n itself included, i.e.,

$$\tau(n) = \sum_{d|n} 1.$$

We use the convention that $\tau(n) = 0$ for $n \leqslant 0$.

Due to Clausen's [9], we have the following identity:

$$\sum_{n=1}^{\infty} \frac{q^n}{1-q^n} = \sum_{n=1}^{\infty} \frac{1+q^n}{1-q^n} q^{n^2}, \quad |q| < 1. \tag{2}$$

In this paper, motivated by these results, we shall prove:

Theorem 1. *For* $|q| < 1$*,*

$$\sum_{n=1}^{\infty} \frac{q^n}{1-q^n} = \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n q^{\binom{n+1}{2}}}{(q;q)_n}, \tag{3}$$

where

$$(a;q)_n = (1-a)(1-aq)\bigl(1-aq^2\bigr)\cdots\bigl(1-aq^{n-1}\bigr)$$

is the q-shifted factorial, with $(a;q)_0 = 1$*.*

Some consequences of this result can be easily derived.