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On quadratic Diophantine equations in four variables and orders associated with lattices II

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The weighted average of the numbers of *primitive solutions* of a quadratic Diophantine equation in four variables connects with the mass of the special orthogonal group of a ternary quadratic form relative to a certain open subgroup, through the mass formula of Shimura. With the determination of suitable group indices, the computation of such a mass can be reduced to that of the mass of the genus of maximal lattices with respect to the ternary form. We determine those indices under several assumptions and provide the numerical examples of the weighted averages for a few positive-definite quaternary quadratic forms.

0. Introduction

Let V be a four-dimensional vector space over a totally real algebraic number field F, φ a totally-positive definite symmetric F-bilinear form on V, and L a maximal lattice in the quadratic space (V, φ) over F. We put

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$$L[q,\mathfrak{b}] = \{x \in V \mid \varphi[x] = q,\ \varphi(x,L) = \mathfrak{b}\}$$

for $0 \neq q \in F$ and a fractional ideal $\mathfrak{b}$ of F, where $\varphi[x] = \varphi(x,x)$. This is the set of *primitive solutions* of the equation $\varphi[x] = q$ in the sense of [6, Introduction I], and it may or may not be empty. Assuming $L[q,\mathfrak{b}] \neq \emptyset$, take an element h of $L[q,\mathfrak{b}]$. We put

$$W = (Fh)^{\perp} = \{x \in V \mid \varphi(x,h) = 0\}$$

and denote by ψ the restriction of φ to W. The special orthogonal group SO^{ψ} of ψ can be regarded as the subgroup $\{\gamma \in SO^{\varphi} \mid h\gamma = h\}$ of the orthogonal group SO^{φ} of φ. Then the mass formula due to Shimura [6, (13.18)] states that

$$\sum_{i\in I} \frac{\#L_i[q,\mathfrak{b}]}{[\Gamma(L_i):1]} = \mathfrak{m}\big(SO^{\psi}, SO^{\psi}_{\mathbf{A}} \cap C(L)\big). \tag{0.1}$$

Here $C(L) = \{\alpha \in SO^{\varphi}_{\mathbf{A}} \mid L\alpha = L\}$, $\Gamma(L) = SO^{\varphi} \cap C(L)$, $\{L_i\}_{i\in I}$ is a complete set of representatives for the SO^{φ}-classes in the SO^{φ}-genus of L, $\mathfrak{m}(SO^{\psi}, SO^{\psi}_{\mathbf{A}} \cap C(L))$ is the mass of SO^{ψ} relative to an open subgroup $SO^{\psi}_{\mathbf{A}} \cap C(L)$ of $SO^{\psi}_{\mathbf{A}}$, and the subscript $\mathbf{A}$ means the adelization of the object in question. We write $R[q,\mathfrak{b}]$ for the quantity of the left-hand side of (0.1) and set $\mathfrak{m}(L) = \sum_{i\in I}[\Gamma(L_i):1]^{-1}$ and $\mathfrak{n}[q,\mathfrak{b}] = \mathfrak{m}(L)^{-1}R[q,\mathfrak{b}]$ for convenience. Then $\mathfrak{n}[q,\mathfrak{b}]$ is determined by $\mathfrak{m}(SO^{\psi}, SO^{\psi}_{\mathbf{A}} \cap C(L))$ and $\mathfrak{m}(L)$. The latter mass does not depend on the choice of L and can be obtained from the exact formula in [4, Theorem 5.8]. Clearly $\mathfrak{n}[q,\mathfrak{b}] = \#L[q,\mathfrak{b}]$ if $\#I = 1$. Our interest in this paper is to compute the mass $\mathfrak{m}(SO^{\psi}, SO^{\psi}_{\mathbf{A}} \cap C(L))$.

Now we consider the integral lattice $L \cap W$ in the orthogonal complement W with respect to ψ. Note that the even Clifford algebra $A^{+}(W)$ of ψ is a quaternion algebra over F. Under some conditions on h, we can take the order $\mathfrak{O}$ associated with $L \cap W$ in $A^{+}(W)$ whose localization is the local order $A^{+}(W_v) \cap S^{+}_{V_v}$ in $A^{+}(W_v)$ with $S^{+}_{V_v}$ defined in [6, Theorem 8.6(ii)] or in [3, (3.4)] at each nonarchimedean prime v of F; see [3, Proposition 3.3(3)]. By [6, (13.13a)] and (1.2) below our mass is then given as follows:

$$\mathfrak{m}\big(SO^{\psi}, SO^{\psi}_{\mathbf{A}} \cap C(L)\big) = \frac{[C(M):\tau(U)]}{[SO^{\psi}_{\mathbf{A}} \cap C(L):\tau(U)]}\mathfrak{m}\big(SO^{\psi}, C(M)\big). \tag{0.2}$$

Here M is a maximal lattice in W with respect to ψ containing $L \cap W$, $U = \prod_{v\in\mathbf{a}} A^{+}(W)^{\times}_{v} \prod_{v\in\mathbf{h}} \mathfrak{O}^{\times}_{v}$, τ is the surjective homomorphism of $A^{+}(W)^{\times}_{\mathbf{A}}$ onto $SO^{\psi}_{\mathbf{A}}$ in (1.2), and $\mathfrak{m}(SO^{\psi}, C(M))$ is the mass of the genus of M. We note that the right-hand side of (0.2) does not depend on the choice of M. Let us denote $\mathfrak{m}(SO^{\psi}, C(M))$ by $\mathfrak{m}(M)$, which can also be derived by [4, Theorem 5.8] if ψ is normalized in the sense of that paper. Therefore the determination of the mass in question can be reduced to the calculation of the group indices in (0.2). This idea of determining the mass was indicated by Shimura [7, §4.9].