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A class group heuristic based on the distribution of 1-eigenspaces in matrix groups ☆

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We propose a modification to the Cohen–Lenstra prediction for the distribution of class groups of number fields, which should also apply when the base field contains non-trivial roots of unity. The underlying heuristic derives from the distribution of 1-eigenspaces in certain generalized symplectic groups over finite rings. The motivation for that heuristic comes from the function field case. We also give explicit formulas for the new predictions in several important cases. These are in close accordance with known data.

1. Introduction

The class group of a number field K is defined as the quotient $\mathrm{Cl}(K) := I_K/P_K$ of the group of fractional ideals I_K by the subgroup of principal ideals P_K in the ring of integers $\mathcal{O}_K$ of K. Despite its importance, not much is known about the behavior of

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these objects. For instance it is still an open question whether there exist infinitely many number fields with trivial class group. In the early 1980's H. Cohen and H.W. Lenstra [6] proposed a heuristic principle, later extended by H. Cohen and J. Martinet [7], which makes predictions on how often a given finite abelian p-group should appear as the p-part $\mathrm{Cl}(K)_p$ of the class group in a specified set of number fields. Only very few instances of these conjectures have been proved (see [5,10] for important recent progress).

In 2008 it was noticed by the second author [13,14] that when the base field K_0 contains pth roots of unity the probabilities postulated by Cohen and Martinet do not match with computational data. In particular this is always the case for $p = 2$. In the absence of theoretical arguments, on the basis of his computational data the second author came up with a conjectural statement [14, Conj. 2.1] describing the behavior of p-parts of class groups in the presence of pth roots of unity in K_0.

Motivated by the works of J. Achter [1,2] who considered the analogous problem on the function field side, we develop a method which can be seen as a theoretical justification for the heuristics of Cohen and Martinet and at the same time for the conjecture in [14]. Our main objects are 1-eigenspaces of elements in what we call the mth symplectic groups $\mathrm{Sp}_{2n}^{(m)}(R)$ over certain finite rings R (see Definition 3.1). The limit for $n \to \infty$ of these eigenspace distributions should then give the right predictions for class group distributions over number fields.

We use the results proved by the first author [4] to compute distributions in these mth symplectic groups (see Theorems 4.4 and 4.6) which allows us to make conjectural predictions (see Conjecture 5.1) about the behavior of p-parts of class groups of number fields. For the case when the base field does not contain pth roots of unity, these specialize to the original Cohen–Lenstra–Martinet predictions (see Example 5.2).

2. Cohen–Lenstra heuristic and roots of unity

In this section we recall the heuristic principle introduced by Cohen and Lenstra to predict the distribution of p-parts of class groups of imaginary quadratic number fields and the generalization to arbitrary number fields proposed by Cohen and Martinet. However, our focus lies on a situation where these predictions seem to fail.

2.1. The Cohen–Lenstra heuristic

Following Cohen and Lenstra [6] we equip finite groups G with their *CL-weight* $\omega(G) := \frac{1}{|\mathrm{Aut}(G)|}$. For integers $q, r \geq 1$ we set

$$(q)_r := \prod_{i=1}^{r} (1 - q^{-i}) \quad \text{and} \quad (q)_\infty := \prod_{i=1}^{\infty} (1 - q^{-i}).$$

For a prime p, let $\mathcal{G}_p$ denote the set of all isomorphism classes of finite abelian p-groups. From [6, Ex. 5.10] we have: