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Hyperquadratic continued fractions in odd characteristic with partial quotients of degree one

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ABSTRACT

In 1986, some examples of algebraic, and nonquadratic, power series over a finite prime field, having a continued fraction expansion with partial quotients all of degree 1 were discovered by W. Mills and D. Robbins. In this note we show that these few examples belong to a very large family of continued fractions for certain algebraic power series over an arbitrary finite field of odd characteristic.

1. Introduction and results

This note deals with continued fractions in fields of power series. For a general account on this matter, the reader can consult W. Schmidt's article [14]. For a wider survey on

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Diophantine approximation in the function field case and full references, the reader may also consult D. Thakur's book [15, Chap. 9]. Let us recall that the pioneer work on the matter treated here, i.e., algebraic continued fractions in power series fields over a finite field, is due to L. Baum and M. Sweet [2].

Let p be a prime number, $q = p^s$ with $s \geq 1$, and let $\mathbb{F}_q$ be the finite field with q elements. We let $\mathbb{F}_q[T]$, $\mathbb{F}_q(T)$ and $\mathbb{F}(q)$ respectively denote the ring of polynomials, the field of rational functions and the field of power series in $1/T$ over $\mathbb{F}_q$, where T is a formal indeterminate. These fields are equipped with the ultrametric absolute value defined by its restriction to $\mathbb{F}_q(T)$: $|P/Q| = |T|^{\deg(P)-\deg(Q)}$, where $|T| > 1$ is a fixed real number. We recall that each irrational (rational) element α of $\mathbb{F}(q)$ can be expanded as an infinite (finite) continued fraction. This will be denoted $\alpha = [a_1, a_2, \ldots, a_n, \ldots]$ where the $a_i \in \mathbb{F}_q[T]$, with $\deg(a_i) > 0$ for $i > 1$, are the partial quotients and the tail $\alpha_i = [a_i, a_{i+1}, \ldots] \in \mathbb{F}(q)$ is the complete quotient. We shall be concerned with infinite continued fractions in $\mathbb{F}(q)$ which are algebraic over $\mathbb{F}_q(T)$.

Regarding Diophantine approximation and continued fractions, a particular subset of elements in $\mathbb{F}(q)$, algebraic over $\mathbb{F}_q(T)$, must be considered. Let $r = p^t$ with $t \geq 0$, we let $\mathcal{H}(r, q)$ denote the subset of irrationals α belonging to $\mathbb{F}(q)$ and satisfying an algebraic equation of the particular form $A\alpha^{r+1} + B\alpha^r + C\alpha + D = 0$, where A, B, C and D belong to $\mathbb{F}_q[T]$. Note that $\mathcal{H}(1, q)$ is simply the set of quadratic irrational elements in $\mathbb{F}(q)$. The union of the subsets $\mathcal{H}(p^t, q)$, for $t \geq 0$, denoted by $\mathcal{H}(q)$, is the set of hyperquadratic power series. For more details and references, the reader may see the introduction of [4]. Even though it contains algebraic elements of arbitrary large degree, this subset $\mathcal{H}(q)$ should be regarded as an analogue, in the formal case, of the subset of quadratic numbers, in the real case. An old and famous theorem, due to Lagrange, gives a characterization of quadratic real numbers as ultimately periodic continued fractions. It is an open problem to know whether another characterization, as particular continued fractions, would be possible for hyperquadratic power series.

The origin of this work is certainly due to a famous example of a cubic power series over $\mathbb{F}_2$, having partial quotients of bounded degrees (1 or 2), introduced in [2]. In a second article [3], Baum and Sweet could characterize all power series in $\mathbb{F}(2)$ having all partial quotients of degree 1 and, among them, those which are algebraic. Underlining the singularity of this context, in [9, p. 5], a different approach could allow to rediscover these particular power series in $\mathbb{F}(2)$. Also in characteristic 2, other algebraic power series over a finite extension of $\mathbb{F}_2$, having all partial quotients of degree 1, were presented (see for instance [10, p. 280]). The case of even characteristic appears to be singular for different reasons. In this note we only consider the case of odd characteristic. Our aim is to show the existence of hyperquadratic continued fractions, in all $\mathbb{F}(q)$'s with odd q, having all partial quotients of degree 1. In $\mathbb{F}(p)$, the first examples were given by Mills and Robbins [12].

Before developing the background of the work presented in this article, we first give an example of such algebraic continued fractions with the purpose of illustrating the