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Inequalities and asymptotic expansions for the gamma function

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ABSTRACT

In this paper, we present new asymptotic expansions of the gamma function. Based on our expansions, we establish some symmetric double inequalities for the gamma function.

1. Introduction

Stirling's formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \quad n \in \mathbb{N} := \{1, 2, \dots\} \tag{1.1}$$

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has many applications in statistical physics, probability theory and number theory. Actually, it was first discovered in 1733 by the French mathematician Abraham de Moivre (1667–1754) in the form

$$n! \sim \text{constant} \cdot \sqrt{n}(n/e)^n$$

when he was studying the Gaussian distribution and the central limit theorem. Afterwards, the Scottish mathematician James Stirling (1692–1770) found the missing constant $\sqrt{2\pi}$ when he was trying to give the normal approximation of the binomial distribution.

Stirling's formula has attracted much interest of many mathematicians and has motivated a large number of research papers concerning various generalizations and improvements (see [3–12,14–19,21–37,39,38,40,41,43]). See also an overview at [20].

A slightly more accurate approximation than Stirling's formula is the Burnside formula [10]:

$$n! \sim \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}}. \tag{1.2}$$

Recently, Mortici [22] published the following simple approximations for $n!$:

$$n! \sim \sqrt{\frac{2\pi}{e}}\left(\frac{n+1}{e}\right)^{n+\frac{1}{2}} \tag{1.3}$$

and

$$n! \sim \sqrt{2\pi e} \cdot e^{-\omega}\left(\frac{n+\omega}{e}\right)^{n+\frac{1}{2}}, \tag{1.4}$$

where $\omega = (3-\sqrt{3}\,)/6$. We find that the formulas (1.1) to (1.4) can be written as

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+O\left(\frac{1}{n}\right)\right), \tag{1.5}$$

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\frac{1}{e^{\frac{1}{2}}}\left(1+\frac{1}{2n}\right)^{n+\frac{1}{2}}\left(1+O\left(\frac{1}{n}\right)\right), \tag{1.6}$$

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\frac{1}{e}\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}\left(1+O\left(\frac{1}{n}\right)\right) \tag{1.7}$$

and

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\frac{1}{e^{\omega}}\left(1+\frac{\omega}{n}\right)^{n+\frac{1}{2}}\left(1+O\left(\frac{1}{n^2}\right)\right), \tag{1.8}$$

respectively. Obviously, the formula (1.8) is better than the formulas (1.5) to (1.7).