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Multiple-correction and faster approximation

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A B S T R A C T

In this paper, we formulate a new *multiple-correction method*. The goal is to accelerate the rate of convergence. In particular, we construct some sequences to approximate the Euler–Mascheroni and Landau constants, which are faster than the classical approximations in literature.

1. Introduction

Euler constant γ was first introduced by Leonhard Euler (1707–1783) in 1734 as the limit of the sequence

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$$\gamma(n) := \sum_{m=1}^{n} \frac{1}{m} - \ln n. \tag{1.1}$$

It is also known as the Euler–Mascheroni constant. There are many famous unsolved problems about the nature of this constant. For example, it is a long-standing open problem if it is a rational number. See e.g. the survey papers or books of Brent and Zimmermann [3], Dence and Dence [13], Havil [20] and Lagarias [21]. A good part of its mystery comes from the fact that the known algorithms converging to γ are not very fast, at least, when they are compared to similar algorithms for π and e.

The sequence $(\gamma(n))_{n\in\mathbb{N}}$ converges very slowly toward γ, like n^{-1}. To evaluate it more accurately, we need to accelerate the convergence. This can be done using the Euler–Maclaurin summation formula, Stieltjes approach, exponential integral methods, Bessel function method, etc. See e.g. Gourdon and Sebah [17].

Up to now, many authors are preoccupied to improve its rate of convergence. See e.g. Chen and Mortici [10], DeTemple [14], Gavrea and Ivan [16], Lu [23,24], Mortici [25], Mortici and Chen [32], Yang [41] and references therein. We list some main results as follows: as $n \to \infty$,

$$\sum_{m=1}^{n} \frac{1}{m} - \ln\left(n + \frac{1}{2}\right) = \gamma + O\left(n^{-2}\right) \quad \left(\text{DeTemple [14], 1993}\right), \tag{1.2}$$

$$\sum_{m=1}^{n} \frac{1}{m} - \ln \frac{n^3 + \frac{3}{2}n^2 + \frac{227}{240} + \frac{107}{480}}{n^2 + n + \frac{97}{240}} = \gamma + O\left(n^{-6}\right) \quad \left(\text{Mortici [25], 2010}\right), \tag{1.3}$$

$$\sum_{m=1}^{n} \frac{1}{m} - \ln \rho(n) = \gamma + O\left(n^{-5}\right) \quad \left(\text{Chen and Mortici [10], 2012}\right), \tag{1.4}$$

where $\rho(n) = 1 + \frac{1}{2n} + \frac{1}{24n^2} - \frac{1}{48n^3} + \frac{23}{5760n^4}$. Recently, Mortici and Chen [32] provided a very interesting sequence

$$\nu(n) = \sum_{m=1}^{n} \frac{1}{m} - \frac{1}{2} \ln\left(n^2 + n + \frac{1}{3}\right) - \left(\frac{-\frac{1}{180}}{(n^2+n+\frac{1}{3})^2} + \frac{\frac{8}{2835}}{(n^2+n+\frac{1}{3})^3} + \frac{\frac{5}{1512}}{(n^2+n+\frac{1}{3})^4} + \frac{\frac{592}{93\,555}}{(n^2+n+\frac{1}{3})^5}\right),$$

and proved

$$\lim_{n\to\infty} n^{12}\left(\nu(n) - \gamma\right) = -\frac{796\,801}{43\,783\,740}. \tag{1.5}$$

Hence, the rate of convergence of the sequence $(\nu(n))_{n\in\mathbb{N}}$ is n^{-12}.