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Zeros of combinations of the Riemann ξ-function on bounded vertical shifts

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ABSTRACT

In this paper we consider a series of bounded vertical shifts of the Riemann ξ-function. Interestingly, although such functions have essential singularities, infinitely many of their zeros lie on the critical line. We also generalize some integral identities associated with the theta transformation formula and some formulae of G.H. Hardy and W.L. Ferrar in the context of a pair of functions reciprocal in Fourier cosine transform.

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Self-reciprocal functions
Theta transformation formula

1. Introduction

The study of the zeros and the 'a-points' of the Riemann zeta-function is of special interest. It is more difficult to locate the zeros or the 'a-points' than to study the value distributions of $\zeta(s)$.

The behavior of $\zeta(s)$ on every vertical line $\sigma = \mathrm{Re}(s) > \frac{1}{2}$ has been studied by Bohr and his collaborators. Let us take the half-plane $\sigma > \frac{1}{2}$, and remove all the points which have the same imaginary part as, and smaller real part than, one of the possible zeros (or the pole) of $\zeta(s)$ in this region. We denote the remaining part of this perforated half-plane by $\mathcal{G}$. Specifically, Bohr and Jessen [3,4] discovered that for $\sigma > \frac{1}{2}$, the limit

$$\lim_{T\to\infty} \frac{1}{T}\mu\big\{\tau \in [0,T] : \sigma + i\tau \in \mathcal{G},\ \log\zeta(\sigma + i\tau) \in \mathcal{R}\big\}$$

exists. Here μ is the Lebesgue measure and $\mathcal{R}$ is any fixed rectangle whose sides are parallel to the axes. Later Voronin [28] provided a generalization of Bohr's denseness result.

For any fixed and distinct numbers $s_1, s_2, \ldots, s_n$ with $\frac{1}{2} < \mathrm{Re}(s_k) < 1$, the set $\{(\zeta(s_1 + it), \ldots, \zeta(s_n + it)) : t \in \mathbb{R}\}$ is dense in $\mathbb{C}^n$. Moreover, for any s with $\frac{1}{2} < \mathrm{Re}(s) < 1$, the set $\{(\zeta(s + it), \ldots, \zeta^{(n)}(s + it)) : t \in \mathbb{R}\}$ is dense in $\mathbb{C}^n$.

Even more striking is Voronin's [29] universality theorem.

Let $0 < r < \frac{1}{4}$ and $g(s)$ be a non-zero analytic function on $|s| \le r$. Then for any $\epsilon > 0$, there exists a positive real number τ such that

$$\max_{|s|\le r}\left|\zeta\left(s + \frac{3}{4} + i\tau\right) - g(s)\right| < \epsilon.$$

Moreover,

$$\liminf_{T\to\infty} \frac{1}{T}\mu\left\{\tau \in [0,T] : \max_{|s|\le r}\left|\zeta\left(s + \frac{3}{4} + i\tau\right) - g(s)\right| < \epsilon\right\} > 0.$$

Concerning Voronin's theorem, Bagchi [2] gave an equivalent condition for the Riemann hypothesis for $\zeta(s)$. He proved in his doctoral thesis that

The Riemann hypothesis is true if, and only if, for any $\epsilon > 0$

$$\liminf_{T\to\infty} \frac{1}{T}\mu\left\{\tau \in [0,T] : \max_{|s|\le r}\left|\zeta\left(s + \frac{3}{4} + i\tau\right) - \zeta(s)\right| < \epsilon\right\} > 0.$$