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Local root numbers, Bessel models, and a conjecture of Guo and Jacquet

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ABSTRACT

Let E/F be a quadratic extension of number fields and D a quaternion algebra over F containing E. Let π_D be a cuspidal automorphic representation of $\mathrm{GL}(n, D)$ and π its Jacquet–Langlands transfer to $\mathrm{GL}(2n)$. Guo and Jacquet conjectured that if π_D is distinguished by $\mathrm{GL}(n, E)$, then π is symplectic and $L(1/2, \pi_E) \neq 0$, where π_E is the base change of π to E. When n is odd, Guo and Jacquet also conjectured a converse. The converse does not always hold when n is even, but we conjecture it holds if and only if certain local root number conditions are satisfied, which is if and only if the corresponding generic representation of the split special orthogonal group $\mathrm{SO}(2n+1)$ has a special E-Bessel model. We use the theta correspondence to relate E-Bessel periods on $\mathrm{SO}(5)$ with $\mathrm{GL}(2, E)$-periods on $\mathrm{GL}(2, D)$, and deduce part of our conjecture when $n = 2$.

1. Introduction

Let F be a number field and $\mathbb{A}$ its adele ring. Let G and H be algebraic groups defined over F with common center Z, and suppose H is a closed subgroup of G. In this paper, a (cuspidal) automorphic representation means an irreducible unitary (cuspidal)

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automorphic representation. We say a cuspidal representation π of $G(\mathbb{A})$ with trivial central character is H-distinguished if the period integral

$$\mathcal{P}_H(\phi) := \int\limits_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})} \phi(h)\, dh$$

defines a nonzero linear form on π.

Let E/F be a quadratic extension of number fields and $X(E:F)$ denote the set of isomorphism classes of quaternion algebras over F which split over E. For $D \in X(E:F)$, let $\mathrm{JL} = \mathrm{JL}_D$ denote the Jacquet–Langlands correspondence of representations from an inner form $\mathrm{GL}(n,D)$ to $\mathrm{GL}(2n)$ defined by Badulescu [2] and Badulescu–Renard [3], and LJ_D denote its inverse. For a cuspidal representation π of $\mathrm{GL}(2n,\mathbb{A})$, π_E denotes the base change of π to $\mathrm{GL}(2n,\mathbb{A}_E)$, and $X(E:F:\pi)$ denotes the set of $D \in X(E:F)$ for which $\pi_D = \mathrm{LJ}_D(\pi)$ exists as a (necessarily cuspidal) representation of $\mathrm{GL}(n,D)(\mathbb{A})$. Note since the matrix algebra M_2 lies in $X(E:F)$, $X(E:F:\pi)$ also contains M_2, in which case $\mathrm{LJ}_{M_2}(\pi) = \pi$. Recall that a cuspidal representation π of $\mathrm{GL}(2n)$ is called symplectic if $L(s,\pi,\Lambda^2)$ has a pole at $s=1$, which is equivalent to being a lift from a generic cuspidal representation of the split orthogonal group $\mathrm{SO}(2n+1)$ by the descent of Ginzburg–Rallis–Soudry (see [15]) or Arthur's trace formula [1].

For each $D \in X(E:F)$, fix an embedding $E \hookrightarrow D$, which gives an embedding $\mathrm{GL}(n,E) \hookrightarrow \mathrm{GL}(n,D)$.

Conjecture 1 *(Guo–Jacquet). (See [18].)*

(1) *Fix $D \in X(E:F)$. Let π_D be a cuspidal representation of $\mathrm{GL}(n,D)(\mathbb{A})$ with trivial central character which has a cuspidal transfer $\pi = \mathrm{JL}(\pi_D)$ to $\mathrm{GL}(2n,\mathbb{A})$. If π_D is $\mathrm{GL}(n,E)$-distinguished, then π is symplectic and $L(1/2,\pi_E) \neq 0$.*

(2) *Suppose n is odd. Let π be a cuspidal representation of $\mathrm{GL}(2n,\mathbb{A})$ with trivial central character. If π is symplectic and $L(1/2,\pi_E) \neq 0$, then there exists a $D \in X(E:F:\pi)$ such that the representation $\pi_D = \mathrm{LJ}_D(\pi)$ of $\mathrm{GL}(n,D)(\mathbb{A})$ is $\mathrm{GL}(n,E)$-distinguished.*

We call part (2) the converse direction of the Guo–Jacquet conjecture, and our goal here is to study the converse direction for n even, though our Conjecture 3 below also partially refines the Guo–Jacquet converse when n is odd.

A few remarks on this conjecture are in order. First, the case $n=1$ was already established by Waldspurger [33]. Waldspurger further proved that, when $n=1$, there is a unique such D in part (2), and by work of Tunnell [31] and Saito [30], this D can be determined uniquely in terms of local root numbers. For $n>1$ odd, it is also reasonable to expect that the D in part (2) is unique and is determined by root numbers. On the other hand, for n even when the converse direction of the Guo–Jacquet conjecture holds, we have the following non-uniqueness conjecture.