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Averaging on thin sets of diagonal forms

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We investigate one-dimensional families of diagonal forms, considering the evolution of the asymptotic formula and error term. We then discuss properties of the average asymptotic formula obtained. The subsequent second moment analysis precipitates an effective means of computing p-adic densities of zeros for large primes p.

1. Introduction

The study of families of integral forms has long been espoused by arithmetic geometers in order to understand fundamental differences in the prevalences of rational points of forms. The idea of counting zeros on average, however, seems to be recent, and has been quite successful in extracting information about typical behaviour in situations where the anomalies have not been classified [2,6,13]. Much of the inspiration for this paper is owed to recent work of Brüdern and Dietmann [6]. By averaging over all coefficients in a range far exceeding the box length B, they demonstrated the Hasse principle for

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almost all degree k diagonal forms in just $3k+2$ variables. Our goal is to see what can be achieved using as little averaging as possible.

Let $k \geqslant 3$ be an integer. Let $h_1, \ldots, h_s$ be polynomials of degree $d > 1$ with integer coefficients, being pairwise relatively prime in $\mathbb{Q}[t]$ and having no integer roots. In the case that k is even, we assume that the leading coefficients of the h_i do not all have the same sign. For positive integers $t \leqslant B^\delta$, where δ is a small positive constant, we first estimate the number $N(B,t)$ of integral solutions $\mathbf{x} \in [-B,B]^s$ to

$$h_1(t)x_1^k + \ldots + h_s(t)x_s^k = 0. \tag{1.1}$$

Suppose that either $s \geqslant 2^k + 1$ and $d\delta < 2^{1-k}$; or $s \geqslant 2k^2 - 2$ and $d\delta < 1/3$. Define

$$\mathfrak{S}(t) = \sum_{q=1}^{\infty} q^{-s} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \prod_{i=1}^{s} \sum_{m=1}^{q} e\big(ah_i(t)m^k/q\big).$$

Theorem 1.1. *There exist positive constants C and ε such that if $B^\varepsilon \ll t \leqslant B^\delta$ then*

$$N(B,t)t^d = C\mathfrak{S}(t)B^{s-k} + O\big(B^{s-k-\varepsilon}\big). \tag{1.2}$$

Theorem 1.1 provides an asymptotic formula for $N(B,t)$, since we know from [7, Chapter 8] that $\mathfrak{S}(t)$ is a positive real number for any integer t. Motivated by the estimate (1.2), we study the asymptotic behaviour of the corresponding weighted average. Let T be a positive integer satisfying $B^\varepsilon \ll T \leqslant B^\delta$.

Theorem 1.2. *There exist positive constants K and ε, independent of T, such that*

$$T^{-1} \sum_{t \leqslant T} N(B,t)t^d B^{k-s} - K \ll B^{-\varepsilon}.$$

Let C and K be as in Theorems 1.1 and 1.2. Put

$$\mathfrak{S} = \sum_{q=1}^{\infty} q^{-1-s} \sum_{t=1}^{q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \prod_{i=1}^{s} \sum_{m=1}^{q} e\big(ah_i(t)m^k/q\big). \tag{1.3}$$

We will see that $K = C\mathfrak{S}$. We will show in Lemma 3.1 that $T^{-1}\sum_{t \leqslant T} \mathfrak{S}(t)$ converges to $\mathfrak{S}$ as $T \to \infty$. If the variance of the singular series were zero then $KB^{s-k}t^{-d}$ would be a good approximation to $N(B,t)$ for almost all t.

Proposition 1.3. *Assume that*

$$\lim_{T \to \infty} T^{-1} \sum_{t \leqslant T} \big(\mathfrak{S}(t) - \mathfrak{S}\big)^2 = 0. \tag{1.4}$$