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Preimages for the Shimura map on Hilbert modular forms

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ABSTRACT

In this article we give a method to construct preimages for the Shimura correspondence on Hilbert modular forms of odd and square-free level. The method relies on the ideas presented for the rational case by Pacetti and Tornaría, and is such that the Fourier coefficients of the preimages constructed can be computed explicitly.

0. Introduction

The Shimura map is a Hecke linear map between half-integral weight modular forms and integral weight ones, introduced in [Shi73] in the classical setting and generalized in [Shi87] to Hilbert modular forms, as well as to the automorphic setting by the work of Waldspurger, Flicker and others. Computing preimages for the Shimura map became an interesting subject after the formulas given by Waldspurger et al., relating the central values of twists of the L-series associated with an integral weight modular form f with

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the Fourier coefficients of a half-integral weight form g mapping to f by the Shimura map. Such formulas have been generalized to the Hilbert setting in [Shi93] and [BM07].

The problem of computing preimages for the Shimura map in the classical setting has been considered, for example, in [Shi75] and [Gro87]. Our method for computing preimages in the Hilbert setting relies on the ideas present in [PT07], which in turn generalize the method of Gross. The preimages are obtained by considering certain ternary theta series associated with ideals in quaternion algebras. The problem of computing these ideals is thus crucial for our method, and has been studied in [DV10] and [PS14].

The correspondence between ideals in quaternion algebras and half-integral weight modular forms has its automorphic counterpart, and was studied by [Wal91] over arbitrary number fields, and in particular in the Hilbert setting. The advantage of our method is that, being more explicit, it allows to compute effectively the Fourier coefficients of the preimages.

In [Xue11] the author also follows the method of Gross for computing half-integral weight Hilbert modular forms to prove a Waldspurger's type formula, but with several restrictions such as working with prime power level and odd class number of the base field, and with no explicit focus on the Shimura correspondence.

We start this article by recalling basic definitions regarding Hilbert modular forms and setting some notation. Some good references for the theory of Hilbert modular forms are Garrett's book [Gar90] and Shimura's article [Shi78].

In the second section, given a totally definite quaternion algebra B and an Eichler order R in it, we define Hecke operators acting on the space $M(R)$ generated by left ideal classes representatives for R, showing that they satisfy properties analogous to those of the Hecke operators on Hilbert modular forms.

In the third section we introduce half-integral weight Hilbert modular forms, following [Shi87]. We state the main properties of the Hecke operators acting on them, and we recall Shimura's correspondence.

In the fourth section we show how certain ternary theta series associated with the left ideal classes of a given order R can be used to produce Hilbert modular forms of parallel weight $3/2$, thus giving a Hecke linear map from the space $M(R)$ to the space of Hilbert modular forms of parallel weight $3/2$.

In the fifth section we use the results from the previous sections to construct preimages of the Shimura map, at least in the case where the level of the modular form is odd and square-free. This is stated in Theorem 5.3, which is our main result. We also state a Waldspurger's type formula relating the Fourier coefficients of the preimages and central values of twisted L-functions.

In the final section we consider the space of Hilbert modular cusp forms over $F = \mathbb{Q}[\sqrt{5}]$, with level $(6+\sqrt{5})$ and parallel weight 2. This space is 1-dimensional, and it is spanned by a newform that corresponds to an elliptic E curve over F. We apply our method to this cusp form to construct a parallel weight $3/2$ modular form in Shimura correspondence with it, and compare its zero coefficients with the ranks of imaginary quadratic twists of E.