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Unique representations of integers using increasing sequences

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A R T I C L E I N F O

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A B S T R A C T

Fraenkel has shown that, if $\langle u_0, u_1, \ldots\rangle$ is any increasing sequence of integers with $u_0 = 1$, any positive integer N can be expressed uniquely in the form $N = \sum_{i=0}^{k} d_i u_i$, where, for $0 \leq j \leq k$, $\sum_{i=0}^{j} d_i u_i < u_{j+1}$. Fraenkel also determined the values that these coefficients d_i can take for certain sequences generated by the linear recurrence $u_n = b_1 u_{n-1} + \ldots + b_m u_{n-m}$. Other authors have proposed variants of these sequences and have determined the values of the d_is for these. This paper extends Fraenkel's definition of his sequences so that $\langle u_1, \ldots, u_{m-1}\rangle$ can be any increasing sequence of integers greater than 1. This definition includes all the above variant sequences and many more that satisfy the above recurrence relation. Again, the values of the coefficients d_i can be determined. Kologlu, Kopp, Miller and Wang and Hamlin and Webb have another unique characterization of coefficients d_i, but we show that these do not in general satisfy the above condition: for $0 \leq j \leq k$, $\sum_{i=0}^{j} d_i u_i < u_{j+1}$.

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1. Introduction

In [1] Fraenkel proved the following:

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Theorem 1.1. *If $1 = u_0 < u_1 < u_2 < \ldots$ is any finite or infinite sequence of integers, any nonnegative integer N has precisely one representation in the system $\langle u_0, u_1, u_2, \ldots \rangle$ of the form $N = \sum_{i=0}^{k} d_i u_i$, where the d_is are nonnegative integers satisfying the following: for $0 \leq j \leq k$, $\sum_{i=0}^{j} d_i u_i < u_{j+1}$.*

We will call such a representation an **F-representation**.

Fraenkel defined a class of sequences in [1], satisfying the recurrence relation

$$u_n = b_1 u_{n-1} + \ldots + b_m u_{n-m}.$$

He had restrictions on the values of the b_is and on those of the m initial u_is, including $u_0 = 1$. The details of this, and his theorem which determined the values the d_is could take, appear in Section 2.

Later authors Petho and Tichy [7], Grabner and Tichy [2] and Grabner, Tichy, Nemes and Petho [3], used different ways of defining the initial values $u_1, \ldots, u_{m-1}$ and also varied the conditions on the b_is. In each case conditions on the d_is could be determined. Details of these variant sequences appear in Section 3.

In Section 4 we show how Fraenkel's definition of his sequences can be altered to allow a much wider range of sequences for which conditions on the d_is can still be determined. We in fact allow any sequence with initial values $u_0 = 1 < u_1 < u_2 < \ldots < u_{m-1}$ that satisfies the recurrence relation.

Kologlu, Kopp, Miller and Wang [5], Miller and Wang [6] and Hamlin and Webb [4] have even weaker conditions on the b_is. Authors of Refs. [5] and [4] also determine the values the d_is can take. We show in Section 5 that, while unique, their representations are not F-representations.

2. Fraenkel's sequences and his theorem on the coefficients

Fraenkel in [1] defined his sequences as follows:

Definition 2.1. For $m \geq 1$, let $b_1 = b_1^{(n)}, b_2, \ldots, b_m$ be integers satisfying $1 \leq b_m \leq b_{m-1} \leq \ldots \leq b_2 \leq b_1^{(n)}$, where $b_2, b_3, \ldots, b_m$ are constants, but $b_1^{(n)}$ may depend on n. Let $u_{-m+1}, u_{-m+2}, \ldots, u_{-1}$ be fixed nonnegative integers. Let $\langle u_0, u_1, \ldots \rangle$ (which we will call an **F-sequence**) be an increasing sequence given by $u_0 = 1$ and, for $n > 0$,

$$u_n = b_1 u_{n-1} + \ldots + b_m u_{n-m}.$$

Note that without the phrase "an increasing sequence" in the definition we could have $b_1^{(n)} = 1$ and $u_{-1} = \ldots = u_{-m+1} = 0$, which gives $u_0 = u_1$.

The definition ensures that, unless m and b_1 are small, $u_1, u_2, \ldots, u_{m-1}$ are quite large.

Example 2.2. $m = 2$, $b_1 = 4$ and $b_2 = 3$. If $u_{-1} = 0$ then $u_1 = 4$; if $u_{-1} = 1$ then $u_1 = 7$.

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