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A Waring–Goldbach type problem for mixed powers

Quanwu Mu [1]

Department of Mathematics, Tongji University, Shanghai, 200092, PR China

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ABSTRACT

Let P_r denote an almost-prime with at most r prime factors, counted according to multiplicity. In this paper, it is proved that for each integer k with $4 \leq k \leq 5$, and for every sufficiently large even integer N satisfying the congruence condition $N \not\equiv 2 \pmod 3$ for $k = 4$, the equation

$$N = x^2 + p_1^2 + p_2^3 + p_3^4 + p_4^4 + p_5^k$$

is solvable with x being an almost-prime P_r and the other variables primes, where $r = 6$ for $k = 4$, and $r = 9$ for $k = 5$. This result constitutes an improvement upon that of R.C. Vaughan.

1. Introduction

Let N, $k_1, k_2, \cdots, k_s$ be natural numbers such that $2 \leq k_1 \leq k_2 \leq \cdots \leq k_s$, $N > s$. Waring problem of mixed powers concerns the representation of N as the form

$$N = x_1^{k_1} + x_2^{k_2} + \cdots + x_s^{k_s}. \tag{1.1}$$

E-mail address: muquanwu@163.com.

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Not very much is known about results of this kind. For historical references the reader should consult section P12 of LeVeque's *Reviews in number theory* and the bibliography in [12].

The circle method of Hardy and Littlewood provides a technique for problems of this sort, but one has to overcome various difficulties not experienced in the pure Waring problem (1.1) with $k_1 = k_2 = \cdots = k_s$. In particular, the choice of the relevant parameters in the definition of major and minor arcs tends to become complicated if a deeper representation problem (1.1) is under consideration.

In 1969, R.C. Vaughan [10] obtained the asymptotic formula for the number of representations of a number as the sum of two squares, one cube and three fourth powers [11]. In view of R.C. Vaughan's result, it is reasonable to conjecture that for every sufficiently large even integer N satisfying the congruence condition $N \not\equiv 2 \pmod 3$ the equation

$$N = p_1^2 + p_2^2 + p_3^3 + p_4^4 + p_5^4 + p_6^4 \tag{1.2}$$

is solvable, where and below the letter p, with or without subscript, always stands for a prime number. The congruence condition $N \not\equiv 2 \pmod 3$ is necessary, because of $p^2 \equiv p^4 \equiv 1 \pmod 3$ and $p^3 \equiv 1$ or $2 \pmod 3$ for $p > 3$. This conjecture is perhaps out of reach at present. It is possible, however, to replace a variable by an almost-prime. Our results are as follows, where an integer with at most r prime factors, counted according to multiplicity, is called an almost-prime P_r, as usual. The proof of our results employs the Hardy–Littlewood method and H. Iwaniec's linear sieve method.

Theorem 1. *For all sufficiently large even integer N with $N \not\equiv 2 \pmod 3$, let $R_4(N)$ denote the number of solutions of the equation*

$$N = x^2 + p_2^2 + p_3^3 + p_4^4 + p_5^4 + p_6^4 \tag{1.3}$$

with x being an almost-prime P_6 and the p_j's primes. Then we have

$$R_4(N) \gg \frac{N^{\frac{13}{12}}}{\log^6 N}.$$

Theorem 2. *For all sufficiently large even integer N, let $R_5(N)$ denote the number of solutions of the equation*

$$N = x^2 + p_2^2 + p_3^3 + p_4^4 + p_5^4 + p_6^5 \tag{1.4}$$

with x being an almost-prime P_9 and the p_j's primes. Then we have

$$R_5(N) \gg \frac{N^{\frac{31}{30}}}{\log^6 N}.$$

We only provide the proof of Theorem 1 in detail, the proof of Theorem 2 follows in a similar manner.