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On finite layers of $\mathbb{Z}_l$-extensions and K_2

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ABSTRACT

Let F denote a number field. We study a relation between the subgroup of elements whose lth roots generate extensions of F which are contained in a $\mathbb{Z}_l$-extension of F and a certain kernel of Milnor's K-group defined by Tate. We prove that both groups can be described in terms of a norm compatible sequence over the cyclotomic $\mathbb{Z}_l$-extension of F.

1. Introduction

Let F be a number field and l an odd prime. An extension field K of F will be called a $\mathbb{Z}_l$-extension if K/F is a Galois extension and the Galois group is isomorphic to the

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additive group of $\mathbb{Z}_l$, the ring of l-adic integers of the field $\mathbb{Q}_l$ of l-adic numbers. We determine such extension fields of F in terms of a norm compatible property over the cyclotomic $\mathbb{Z}_l$-extension of F. This shows that an information on an arithmetic property of the cyclotomic $\mathbb{Z}_l$-extension of F determines all $\mathbb{Z}_l$-extensions of F.

Bertrandias and Payan proved in their paper [3] a theorem which determines the first layers of $\mathbb{Z}/l^n\mathbb{Z}$-extensions of a number field in terms of a certain norm group over the cyclotomic $\mathbb{Z}/l^n\mathbb{Z}$-extension of the ground field. The main theorem in this paper is a natural generalization of this.

More precisely, the first layers of the inverse limit over the cyclotomic $\mathbb{Z}/l^n\mathbb{Z}$-extension play a role to find the first layers of $\mathbb{Z}_l$-extensions in the same way that the norm groups are used in [3] to find the first layers of $\mathbb{Z}/l^n\mathbb{Z}$-extensions.

For a positive integer $s > 0$, let ζ_s denote a primitive sth root of unity in a fixed algebraic closure $\overline{F}$ of F such that $\zeta_s^l = \zeta_{sl^{-1}}$ for any $l|s$. Let $m \geq 0$ denote the maximum number with $\zeta_{l^m} \in F$. Let $\tilde{F}_\infty$ denote the composite of all $\mathbb{Z}_l$-extensions of F inside $\overline{F}$. Let w be a fixed integer with $0 < w \leq m$ when $m > 0$ and $w = 1$ for $m = 0$. Let Θ_F^w denote the set of all elements α in $F^\times$ such that the field generated by l^wth roots of α is contained in a $\mathbb{Z}_l$-extension of F,

$$\Theta_F^w = \{\alpha \in F^\times \mid F(\alpha^{1/l^w}) \subset \tilde{F}_\infty\}.$$

For a notational convenience, we write $\Theta_F = \Theta_F^1$.

Let $\mathbb{Q}(\mu_{l^\infty}) = \bigcup_{n \geq m} \mathbb{Q}(\mu_{l^n})$ and let $\mathbb{Q}_\infty$ denote the unique $\mathbb{Z}_l$ extension of $\mathbb{Q}$ which is the fixed field of the torsion subgroup of the Galois group $G(\mathbb{Q}(\mu_{l^\infty})/\mathbb{Q})$ in $\mathbb{Q}(\mu_{l^\infty})$. We denote by $F_\infty^{\mathrm{cyc}} = F\mathbb{Q}_\infty$ the cyclotomic $\mathbb{Z}_l$-extension of F in $\overline{F}$ and F_n its unique subfield of degree l^{n-m} over F for $n \geq m$ and $F_n = F$ for $n < m$. We let

$$\varprojlim_{n \geq m} F_n^\times / T F_n^{\times l^{n-m}} \cdot F_n^{\times l^n} \cdot F^{\times l^m} \quad \text{and} \quad \varprojlim_{n \geq m} F_n^\times / T F_n^{\times l^{m-1}} \cdot (T - l^m) F_n^\times$$

denote the inverse limit of $F_n^\times / T F_n^{\times l^{n-m}} \cdot F_n^{\times l^n} \cdot F^{\times l^m}$ and $F_n^\times / T F_n^{\times l^{m-1}} \cdot (T - l^m) F_n^\times$ respectively over $n \geq m$ with respect to the field theoretic norm maps. Let $\pi_{w,1}$ and $\pi_{w,2}$ denote the natural projections

$$\pi_{w,1} : \varprojlim_{n \geq m} F_n^\times / T F_n^{\times l^{n-m}} \cdot F_n^{\times l^n} \cdot F^{\times l^m} \longrightarrow F^\times / F^{\times l^w}$$

$$\pi_{w,2} : \varprojlim_{n \geq m} F_n^\times / (T - l^m) F_n^\times \cdot F^{\times l^{m-1}} \longrightarrow F^\times / F^{\times l^w}$$

defined as $\pi_{w,1}(a_n \bmod T F_n^{\times l^{n-m}} \cdot F_n^{\times l^n} \cdot F^{\times l^m}) = N_{m+1}(a_{m+1}) \bmod F^{\times l^w}$ and

$$\pi_{w,2}(a_n \bmod F_n^\times / (T - l^m) F_n^\times \cdot F^{\times l^{m-1}}) = N_{m+1}(a_{m+1}) \bmod F^{\times l^w}$$

where N_n denotes the field theoretic norm map from F_n to F.