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# Sign changes of Fourier coefficients of Hilbert modular forms

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Sign changes of Fourier coefficients of various modular forms have been studied. In this paper, we analyze some sign change properties of Fourier coefficients of Hilbert modular forms, under the assumption that all the coefficients are real. The quantitative results on the number of sign changes in short intervals are also discussed.



## 1. Introduction

The Fourier coefficients of modular forms are interesting objects because of their nice arithmetic and algebraic properties. It is easy to see that the Fourier coefficients of a cusp

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form for $\Gamma_0(N)$ change signs infinitely often if the coefficients are all real numbers. In fact, the signs of the Fourier coefficients determine a cusp form. The signs of the Fourier coefficients of cusp forms were first studied by M. Ram Murty in [9]. After that there has been more extensive study of the Fourier coefficients of other kinds of automorphic forms. In this article, we first prove a sign change result in the case of Hilbert modular forms. More precisely, we prove the following.

**Theorem 1.1.** *Let* $\mathbf{f}$ *be a Hilbert cusp form of weight* $k = (k_1, \ldots, k_n)$ *and level* $\mathfrak{n}$*, and let* $C(\mathfrak{m})$ *be a Fourier coefficient of* $\mathbf{f}$ *at each integral ideal* $\mathfrak{m}$*. If* $\{C(\mathfrak{m})\}$ *are all real, then there are infinitely many sign changes on* $\{C(\mathfrak{m})\}$*.*

Here, $n$ is the extension degree of the base field. All the setting is precisely described in Section 2.

Next question which naturally arises in the case of cusp forms is to determine a bound for the first sign change to occur in the sequence of Fourier coefficients. Bounds have been obtained by Kohnen and Sengupta [7], Iwaniec, Kohnen and Sengupta [6], and Choie and Kohnen [4]. More generally, Qu [11] has obtained a similar kind of bound for the first sign change of the coefficients for the automorphic $L$-function attached to an irreducible unitary cuspidal representation for $\mathrm{GL}_m(\mathbb{A}_\mathbb{Q})$, under the assumption that all the coefficients are real. Thus it naturally comes to our mind to get a bound of similar kind in the case of Hilbert modular forms. In our next result which is stated below, we get an affirmative answer.

**Theorem 1.2.** *Let* $\mathbf{f}$ *be a primitive Hilbert cusp form of weight* $k = (k_1, \ldots, k_n)$*, level* $\mathfrak{n}$ *and with the trivial character. Write* $\{C(\mathfrak{m})\}$ *for Fourier coefficients of* $\mathbf{f}$*, and let* $Q_\mathbf{f}$ *be the analytic conductor of* $\mathbf{f}$*. Then there exists an integral ideal* $\mathfrak{m}$ *with*

$$\mathrm{N}(\mathfrak{m}) \ll_{n,\epsilon} Q_\mathbf{f}^{1+\epsilon}$$

*such that* $C(\mathfrak{m}) < 0$*.*

Finally, we consider the behavior of the signs of the coefficients in short intervals $(x, 2x)$ for sufficiently large $x$. Namely, we prove the following quantitative result for the number of sign changes in the interval $(x, 2x)$.

**Theorem 1.3.** *Let* $\mathbf{f}$ *be a primitive Hilbert cusp form of weight* $k = (k_1, \ldots, k_n)$*, full level, and with the trivial character. Assume that the weight satisfies the following congruence property:* $k_1 \equiv \cdots \equiv k_n \equiv 0 \bmod 2$*. For each integral ideal* $\mathfrak{m}$ *of* $F$*, let* $C(\mathfrak{m})$ *be a Fourier coefficient of* $\mathbf{f}$ *at* $\mathfrak{m}$*. Then, for any* $r$ *with* $\frac{4n-1}{4n+1} < r < 1$*, at least one sign change for* $\{C(\mathfrak{m})\}$ *occurs with* $\mathrm{N}(\mathfrak{m}) \in (x, x + x^r]$*.*

This follows from a recent work of Meher and Murty [8], together with a result of Chandrasekharan and Narasimhan [3] and Ramanujan conjecture for Hilbert modular