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Arithmetic of octahedral sextics

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Given a quartic field with S_4 Galois group, we relate its ramification to that of the non-Galois sextic subfields of its Galois closure, and we construct explicit generators of these sextic fields from that of the quartic field, and vice versa. This allows us to recover examples of S_4-sextic fields of Cohen and of Tate unramified outside 229, and to easily determine the tame part of the conductor of an octahedral Artin representation. We study class number divisibility arising from S_4-quartics whose discriminants are odd and square-free, we explicitly construct infinitely many S_4-quartics whose discriminants are -1 times a square, and experimental data suggest two surprising conjectures about S_4-quartic fields over $\mathbf{Q}$ unramified outside one finite prime.

1. Introduction

Cohen [3, §6.3.7] notes that the sextic polynomials

$$f_1(x) = x^6 - 4x^2 - 1, \qquad \text{poly disc } 2^6 229^2, \qquad \text{field disc } 229^2$$

$$f_2(x) = x^6 - 3x^5 + 6x^4 - 7x^3 + 2x^2 + x - 4, \qquad \text{poly disc } 229^3, \qquad \text{field disc } 229^3$$

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are irreducible over $\mathbf{Q}$ and have Galois group S_4, the symmetric group on four letters. The discriminant of f_1 is a square and that of f_2 is not, so S_6 has two[1] transitive subgroups isomorphic to S_4, one of which is contained in A_6 and the other one is not. Note that $f_1(x) = -\lambda(-x^2)$ where $\lambda = x^3 - 4x + 1$, and that $64f_2(\frac{x+1}{2}) = x^6 + 9x^4 - 37x^2 - 229$ is $\bar{\lambda}(x^2)$ for a cubic $\bar{\lambda}$ over $\mathbf{Q}$ with $-64\lambda(\frac{-x-3}{4}) = \bar{\lambda}(x)$, so the sextic fields defined by f_1 and f_2 contain the same (isomorphism class of) cubic field of discriminant 229 defined by λ.

Sextic fields unramified outside 229 also appear in classical examples of Artin representations. Let γ be a root of λ. Tate [15, p. 251] shows that the minimal polynomials

$$g_1 := \min_{\mathbf{Q}}(\sqrt{-3+8\gamma}\,) = x^6 + 9x^4 - 229x^2 - 229, \quad \text{field disc } 229^3,$$
$$h_1 := \min_{\mathbf{Q}}\big(\sqrt{4-3\gamma^2}\,\big) = x^6 + 12x^4 - 229, \quad \text{field disc } 229^3,$$

are irreducible over $\mathbf{Q}$, have S_4 Galois groups, and the projective Artin representations defined by their splitting fields lift to ordinary odd, 2-dimensional Artin representations ρ_1, ρ_2 of level 229 such that ρ_1 is not isomorphic to ρ_2 or its dual. Using the computer algebra system `PARI-GP` we check that f_1, f_2, g_1, h_1 define pairwise non-isomorphic fields unramified outside 229. The starting point of this paper is to find all such sextic fields, and to understand these examples by putting them in a wider context. We will relate the discriminant of an S_4-quartic field K_4 to that of the non-Galois sextic subfields of its Galois closure, and we will study S_4-quartic fields whose discriminants are square-free, -1 times a perfect square, and prime-powers, respectively.

We begin with a general setup. Let k be a field of characteristic $\neq 2$, and let K_{24}/k be a degree 24 Galois extension with $\mathrm{Gal}(K_{24}/k) \simeq S_4$. There is one conjugacy class of subgroups in S_4 isomorphic to each of $\mathbf{Z}/4$, D_3 (the dihedral group of order 6) and D_4, respectively. It also has one conjugacy class of *non-normal* subgroups isomorphic to $\mathbf{Z}/2 \times \mathbf{Z}/2$ (generated by any pair of disjoint 2-cycles). Fix a subgroup from each one of these conjugacy classes, and consider the fixed field by each of these representative subgroups:

K_3: fixed field by D_4;
K_4: fixed field by D_3;
K_6: fixed field by $\mathbf{Z}/4$;
K_6': fixed field by a non-normal $\mathbf{Z}/2 \times \mathbf{Z}/2$.

The D_4 subgroups of S_4 are the Sylow 2-subgroups, so they contain conjugates of $\mathbf{Z}/4$ and (normal and non-normal) $\mathbf{Z}/2 \times \mathbf{Z}/2$. This can also be seen from the following explicit model of a D_4 subgroup in S_4 which we will make use of later on:

$$\big\{(1234),\ (1432),\ (13)(24),\ (),\ (13),\ (24),\ (12)(34),\ (14)(23)\big\}. \tag{1}$$

[1] There are exactly four conjugacy classes of S_4 subgroups in S_6: Two of them are transitive and two are not, and in each case exactly one of the two S_4 is contained in A_6 (cf. [19]). We will not make use of these remarks in the rest of the paper.