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Transcendence of generalized Euler–Lehmer constants

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A B S T R A C T

In this article, we study the arithmetic properties of generalized Euler–Lehmer constants. We show that these infinite family of numbers are transcendental with at most one exception. This result generalizes a recent result of Murty and Zaytseva.

1. Introduction

Throughout the paper, p denotes a prime number. For a non-empty finite set of primes Ω and $P = \prod_{p\in\Omega} p$, define an arithmetic function 1_Ω by

$$1_\Omega(n) := \begin{cases} 1 & \text{if } (n, P) = 1, \\ 0 & \text{otherwise} \end{cases}$$

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and the generalized Euler constant by

$$\gamma(\Omega) := \lim_{x \to \infty} \left(\sum_{n \le x} \frac{1_\Omega(n)}{n} - \delta_\Omega \log x \right).$$

Here

$$\delta_\Omega := \lim_{x \to \infty} \frac{1}{x} \sum_{n \le x} 1_\Omega(n)$$
$$= \prod_{p \in \Omega} \left(1 - \frac{1}{p} \right).$$

Further, when $\Omega = \phi$, we define $1_\Omega(n) := 1$ for all $n \in \mathbb{N}$, $\delta_\Omega := 1$ and hence $\gamma(\Omega) = \gamma$, Euler's constant. These generalized Euler's constants were introduced and studied by Diamond and Ford in [3]. In a recent paper, Murty and Zaytseva [9] showed that as Ω varies over all finite subsets of primes, all $\gamma(\Omega)$ are transcendental with at most one exception.

In this article, we introduce and study the possible transcendental nature of generalized Euler–Lehmer constants. The Euler–Lehmer constants were introduced by Lehmer [5] in 1975. The transcendence of Euler–Lehmer constants and their p-adic analogues were studied in [2,7,8]. For an exhaustive account of Euler's constant, see the recent article by Lagarias [4].

For natural numbers a and $q > 1$ with $(a, q) = 1$ and for a finite set of primes Ω not containing any prime factors of q, we define the generalized Euler–Lehmer constants by

$$\gamma(\Omega, a, q) := \lim_{x \to \infty} \left(\sum_{\substack{n \le x \\ n \equiv a \bmod q}} \frac{1_\Omega(n)}{n} - \delta_\Omega \frac{\log x}{q} \right).$$

In this context, we prove the following theorems:

Theorem 1. *Let a and $q > 1$ be natural numbers with $(a, q) = 1$ and γ be Euler's constant. Then*

$$\gamma(\Omega, a, q) - \delta_\Omega \frac{\gamma}{q}$$

is transcendental.

Theorem 2. *Let a and $q > 1$ be natural numbers with $(a, q) = 1$ and S be the set of prime divisors of q. Also let*

$$U := \{\Omega \mid \Omega \text{ is a finite set of primes}, \Omega \cap S = \phi\}.$$

Then the set $T := \{\gamma(\Omega, a, q) \mid \Omega \in U\}$ is infinite and has at most one algebraic element.