Journal of Number Theory 145 (2014) 362–387

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

On simultaneous diophantine approximations to $\zeta(2)$ and $\zeta(3)$

Simon Dauguet [a], Wadim Zudilin [b,*,1]

[a] Université Paris-Sud, Laboratoire de Mathématiques d'Orsay, Orsay Cedex, F-91405, France

[b] School of Mathematical and Physical Sciences, The University of Newcastle, Callaghan, NSW 2308, Australia

ARTICLE INFO

Article history:
Received 9 December 2013
Received in revised form 12 June 2014
Accepted 12 June 2014
Available online 12 August 2014
Communicated by David Goss

MSC:
primary 11J82
secondary 11J72, 33C20

Keywords:
Irrationality
Diophantine approximation
Irrationality exponent
Value of Riemann's zeta function
Hypergeometric series
Hypergeometric integral
Recurrence equation
Algorithm of creative telescoping

ABSTRACT

We present a hypergeometric construction of rational approximations to $\zeta(2)$ and $\zeta(3)$ which allows one to demonstrate simultaneously the irrationality of each of the zeta values, as well as to estimate from below certain linear forms in 1, $\zeta(2)$ and $\zeta(3)$ with rational coefficients. We then go further to formalize the arithmetic structure of these specific linear forms by introducing a new notion of (simultaneous) diophantine exponent. Finally, we study the properties of this newer concept and link it to the classical irrationality exponent and its generalizations given recently by S. Fischler.

* Corresponding author.
E-mail addresses: simon.dauguet@math.u-psud.fr (S. Dauguet), wzudilin@gmail.com (W. Zudilin).

1 The second author is supported by the Australian Research Council, grant DP140101186.

http://dx.doi.org/10.1016/j.jnt.2014.06.007

1. Introduction

It is known that the Riemann zeta function $\zeta(s)$ takes irrational values at positive even integers. This follows from Euler's evaluation $\zeta(s)/\pi^s \in \mathbb{Q}$ for $s = 2, 4, 6, \dots$ and from the transcendence of π. Less is known about the values of $\zeta(s)$ at odd integers $s > 1$. Apéry was the first to establish the irrationality of such a zeta value $\zeta(s)$: he proved [Apé79] in 1978 that $\zeta(3)$ is irrational. The next major step in the direction was made by Ball and Rivoal [BR01] in 2000: they showed that there are infinitely many odd integers at which Riemann zeta function is irrational. Shortly after, Rivoal demonstrated [Riv02] that one of the nine numbers $\zeta(5), \zeta(7), \dots, \zeta(21)$ is irrational, while the second author [Zud01] reduced the nine to four: he proved that at least one of the four numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$ and $\zeta(11)$ is irrational.

Already in 1978, Apéry constructs linear forms in 1 and $\zeta(2)$, as well as in 1 and $\zeta(3)$, with integer coefficients that produce the irrationality of the two zeta values in a quantitative form: the constructions imply upper bounds $\mu(\zeta(2)) < 11.850878\dots$ and $\mu(\zeta(3)) < 13.41782\dots$ for the irrationality measures. Recall that the irrationality exponent $\mu(\alpha)$ of a real irrational α is the supremum of the set of exponents μ for which the inequality $|\alpha - p/q| < q^{-\mu}$ has infinitely many solutions in rationals p/q. Hata improves the above mentioned results to $\mu(\zeta(2)) < 5.687$ in [Hat95, Addendum] and to $\mu(\zeta(3)) < 7.377956\dots$ in [Hat00]. Further, Rhin and Viola study a permutation group related to $\zeta(2)$ in [RV96] and show that $\mu(\zeta(2)) < 5.441243$. They later apply their new permutation group arithmetic method to $\zeta(3)$ as well, to prove the upper bound $\mu(\zeta(3)) < 5.513891$. In an attempt to unify the achievements of Ball–Rivoal and of Rhin–Viola, the second author re-interpreted the constructions using the classical theory of hypergeometric functions and integrals [Zud04]. In his recent work [Zud14], he uses the permutation group arithmetic method and a hypergeometric construction, closely related to the one in this paper, to sharpen the earlier irrationality exponent of $\zeta(2)$ to $\mu(\zeta(2)) \le 5.09541178\dots$.

In this paper, we construct simultaneous rational approximations to both $\zeta(2)$ and $\zeta(3)$ using hypergeometric tools, and establish from them a lower bound for $\mathbb{Q}$-linear combinations of 1, $\zeta(2)$ and $\zeta(3)$ under some strong divisibility conditions on the coefficients. Namely, we prove

Theorem 1. *Let η and ε be positive real numbers. For m sufficiently large with respect to ε and η, let $(a_0, a_1, a_2) \in \mathbb{Q}^3 \setminus \{\mathbf{0}\}$ be such that*

(i) *$D_m^2 D_{2m} a_0 \in \mathbb{Z}$, $D_m a_1 \in \mathbb{Z}$ and $\frac{D_{2m}}{D_m} a_2 \in \mathbb{Z}$, where D_m denotes the least common multiple of $1, 2, \dots, m$; and*
(ii) *$|a_0|, |a_1|, |a_2| \le e^{-(\tau_0+\varepsilon)m}$ hold with $\tau_0 = 0.899668635\dots$.*

Then $|a_0 + a_1\zeta(2) + a_2\zeta(3)| > e^{-(s_0+\eta)m}$ with $s_0 = 6.770732145\dots$.

Theorem 1 contains the irrationality of both $\zeta(2)$ and $\zeta(3)$, because $\tau_0 < 1$. Namely, taking