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The growth rate of the partial quotients in a class of continued fractions with parameters

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ABSTRACT

Let $\epsilon : \mathbb{N} \to \mathbb{R}$ be a parameter function satisfying the condition $\epsilon(k)+k+1>0$ and let $T_\epsilon : (0,1] \to (0,1]$ be a transformation defined by

$$T_\epsilon(x) = \frac{-1+(k+1)x}{1+\epsilon(k)-k\epsilon(k)x} \quad \text{for } x \in \big(1/(k+1), 1/k\big].$$

Under the algorithm T_ϵ, every $x \in (0,1]$ is attached an expansion, called generalized continued fraction expansion with parameters by F. Schweiger [3]. Define the sequence $\{k_n(x)\}_{n\geq 1}$ of the partial quotients of x by $k_1(x) = \lfloor 1/x \rfloor$ and $k_n(x) = k_1(T_\epsilon^{n-1}(x))$ for every $n \geq 2$. It is clear that under the condition satisfied by the parameter function ϵ, $k_{n+1}(x) \geq k_n(x)$ for all $n \geq 1$. In this paper, we consider the size of the set given by

$$E_\epsilon(\alpha) := \big\{x \in (0,1] : k_{n+1}(x) \geq k_n(x)^\alpha \text{ for all } n \geq 1\big\}$$

for any $\alpha \geq 1$. We show that

$$\dim_H E_\epsilon(\alpha) = \begin{cases} \frac{1}{\alpha}, & \text{when } \epsilon(k) \equiv \epsilon_0 \text{ (constant)}; \\ \frac{1}{\alpha-\beta+1}, & \text{when } \epsilon(k) \sim k^\beta \text{ and } \alpha \geq \beta \geq 1; \\ 1, & \text{when } \epsilon(k) \sim k^\beta \text{ and } \alpha < \beta. \end{cases}$$

where $\dim_H$ denotes the Hausdorff dimension. The first result generalizes a result of J. Wu [5] who considered the case when $\epsilon \equiv 0$ (i.e., Engel expansion).

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1. Introduction

In 2003, F. Schweiger [3] introduced a class of continued fraction with parameters, called generalized continued fraction (GCF_ϵ), which is induced by the transformation $T_\epsilon : (0, 1] \to (0, 1]$

$$T_\epsilon(x) := \frac{-1 + (k + 1)x}{1 + \epsilon(k) - k\epsilon(k)x} \quad \text{for } x \in B(k) := \big(1/(k + 1), 1/k\big]. \tag{1.1}$$

where the parameter $\epsilon : \mathbb{N} \to \mathbb{R}$ satisfies

$$\epsilon(k) + k + 1 > 0. \tag{1.2}$$

Define the partial quotients $k_1, k_2, \cdots$ of GCF_ϵ expansion of $x \in (0, 1]$ as

$$k_1 = k_1(x) := \left\lfloor \frac{1}{x} \right\rfloor, \quad \text{and} \quad k_n = k_n(x) := k_1\big(T_\epsilon^{n-1}(x)\big) \quad \text{for } n \geq 2.$$

Then it follows from the algorithm (1.1) that [3]

$$x = \frac{A_n + B_n T_\epsilon^n(x)}{C_n + D_n T_\epsilon^n(x)} \quad \text{for all } n \geq 1,$$

where the numbers A_n, B_n, C_n, D_n are given by the following recursive relations:

$$\begin{pmatrix} C_0 & D_0 \\ A_0 & B_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} C_n & D_n \\ A_n & B_n \end{pmatrix} = \begin{pmatrix} C_{n-1} & D_{n-1} \\ A_{n-1} & B_{n-1} \end{pmatrix} \begin{pmatrix} k_n + 1 & k_n\epsilon(k_n) \\ 1 & 1 + \epsilon(k_n) \end{pmatrix}, \quad n \geq 1. \tag{1.3}$$

From the condition shared by the parameter $\epsilon(k)$ (1.2), it is clear that

$$k_{n+1}(x) \geq k_n(x) \quad \text{for all } n \geq 1, \tag{1.4}$$

i.e. the partial quotients sequence of x is non-decreasing.

F. Schweiger [3] studied the arithmetic and ergodic properties of this new continued fraction expansion and found surprisingly that different choice of the parameter function ϵ leads to different stochastic properties of the partial quotients. For the growth rate of the partial quotients, T. Zhong [6] showed that when $-1 < \epsilon(k) \leq 1$, for almost all $x \in (0, 1]$,