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On the quantitative subspace theorem

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ABSTRACT

In 2008, Evertse and Ferretti stated a quantitative version of the Subspace Theorem for a projective variety with higher degree polynomials instead of linear forms. Our goal is to generalize their results.

1. Introduction

1.1. We first recall some notation (see [6]). In this paper, by a projective subvariety of $\mathbb{P}^N$, we mean a geometrically irreducible Zariski-closed subset of $\mathbb{P}^N$. For a Zariski-closed subset X of $\mathbb{P}^N$ and for a field Ω, we denote by $X(\Omega)$ the set of Ω-rational points of X.

All number fields considered in this paper are contained in a given algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$. Let K be a number field and denote by G_K the Galois group of $\bar{\mathbb{Q}}$ over K. For $x = (x_0, \ldots, x_N) \in \bar{\mathbb{Q}}^{N+1}$, $\sigma \in G_K$, we write

$$\sigma(x) = \big(\sigma(x_0), \ldots, \sigma(x_N)\big).$$

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Let $\mathcal{O}_K$ denote the ring of integers of K. We have a canonical set M_K of places (or absolute values) of K consisting of one place for each prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$, one place for each real embedding $\sigma : K \longrightarrow \mathbb{R}$, and one place for each pair of conjugate embedding $\sigma, \bar{\sigma} : K \longrightarrow \mathbb{C}$. For $v \in M_K$, let K_v denote the completion of K with respect to v. We normalize our absolute values so that $|p|_v = p^{-[K_v:\mathbb{Q}_p]/[K:\mathbb{Q}]}$ if v corresponds to $\mathfrak{p}$ and $\mathfrak{p}|p$ (in which case we say that v is non-Archimedean), and $|x|_v = |\sigma(x)|^{[K_v:\mathbb{R}]/[K:\mathbb{Q}]}$ if v corresponds to an embedding σ (in which case we say that v is Archimedean). Denote by M_K^∞ (resp. M_K^0) the set of Archimedean (resp. non-Archimedean) places. We also note that, if v is a place of K and w is a place of a field extension L of K, then we say that w lies above v (or v lies below w), denoted by $w|v$, if w and v define the same topology on K. These absolute values satisfy the product formula

$$\prod_{v \in M_K} |x|_v = 1 \quad \text{for } x \in K^*.$$

For each $x = [x_0 : ... : x_N] \in K^{N+1}$, we put

$$\|x\|_v := \max\bigl(|x_0|_v, \ldots, |x_N|_v\bigr),$$

for $v \in M_K$. Then the absolute logarithmic height of x is defined by

$$h(x) = \log\biggl(\prod_{v \in M_K} \|x\|_v\biggr).$$

By the product formula, this is well-defined in $\mathbb{P}^N(K)$. Moreover, $h(x)$ doesn't depend on the choice of the particular number field K containing $x_0, \ldots, x_N$. Thus, this function h gives rise to a height on $\mathbb{P}^N(\bar{\mathbb{Q}})$.

For every $v \in M_K$, we choose an extension of $|.|_v$ to $\bar{\mathbb{Q}}$ (this amounts to extending $|.|_v$ to the algebraic closure $\bar{K}_v$ of K_v and choosing an embedding of $\bar{\mathbb{Q}}$ into $\bar{K}_v$). Further, for $v \in M_K$, $x = (x_0, \ldots, x_N) \in \bar{\mathbb{Q}}^{N+1}$, we put

$$\|x\|_v := \max\bigl(|x_0|_v, \ldots, |x_N|_v\bigr).$$

Given a system $f_0, \ldots, f_m$ of polynomials with coefficients in $\bar{\mathbb{Q}}$, we define

$$h(f_0, \ldots, f_m) := h(a),$$

where a is a vector consisting of the non-zero coefficients of $f_0, \ldots, f_m$. Further by $K(f_0, \ldots, f_m)$, we denote the extension of K generated by the coefficients of $f_0, \ldots, f_m$. The height of a projective subvariety X of $\mathbb{P}^N$ defined over $\bar{\mathbb{Q}}$ is defined by

$$h(X) := h(F_X),$$

where F_X is the Chow form of X (see Paragraph 2.2 below).