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On a continued fraction of order twelve and new Eisenstein series identities

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ABSTRACT

We prove two identities associated with Ramanujan's continued fraction of order 12. We further establish several Eisenstein series identities associated with Ramanujan's continued fraction of order 12.

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1. Introduction

Throughout this paper, we assume that $|q| < 1$ and use the standard product notation

$$(a;q)_0 := 1, \qquad (a;q)_n := \prod_{j=0}^{n-1}\bigl(1-aq^j\bigr) \quad \text{and} \quad (a;q)_\infty := \prod_{n=0}^{\infty}\bigl(1-aq^n\bigr).$$

For convenience, we sometimes use the multiple q-shifted factorial notation, which is defined as

$$(a_1,a_2,\ldots,a_m;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty.$$

The well-known Rogers–Ramanujan functions $G(q)$ and $H(q)$ are defined by

$$G(q) := \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n} \quad \text{and} \quad H(q) := \sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q;q)_n},$$

which satisfy the famous Rogers–Ramanujan identities [15, pp. 214–215], [18]

$$G(q) = \frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty} \quad \text{and} \quad H(q) = \frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}. \tag{1.1}$$

The famous Rogers–Ramanujan continued fraction is defined by

$$R(q) := \frac{q^{\frac{1}{5}}}{1}\;{}_{+}\;\frac{q}{1}\;{}_{+}\;\frac{q^2}{1}\;{}_{+}\;\frac{q^3}{1}\;{}_{+}\;\cdots,$$

and the infinite product representation for $R(q)$ is given by [15, p. 377]

$$R(q) = q^{\frac{1}{5}}\frac{H(q)}{G(q)} = q^{1/5}\frac{(q,q^4;q^5)_\infty}{(q^2,q^3;q^5)_\infty}.$$

Ramanujan eventually found several generalizations and ramifications of $R(q)$ which can be found in his notebooks [16] and "Lost Notebook" [17]. Motivated by these, recently, Z.-G. Liu [12] and H.H. Chan, S.H. Chan and Liu [5] have established several new identities associated with the Rogers–Ramanujan continued fraction $R(q)$ and also derived a new Eisenstein series identity involving $R(q)$.

The celebrated Ramanujan–Göllnitz–Gordon continued fraction is defined by [16]

$$K(q) := \frac{q^{1/2}}{1+q}\;{}_{+}\;\frac{q^2}{1+q^3}\;{}_{+}\;\frac{q^4}{1+q^5}\;{}_{+}\;\cdots,$$

and the well-known Göllnitz–Gordon functions are defined by

$$S(q) := \sum_{n=0}^{\infty}\frac{(-q;q^2)_n}{(q^2;q^2)_n}q^{n^2} \quad \text{and} \quad T(q) := \sum_{n=0}^{\infty}\frac{(-q;q^2)_n}{(q^2;q^2)_n}q^{n^2+2n}.$$