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Hadamard products of algebraic functions

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ABSTRACT

Allouche and Mendès France [1] have defined the grade of a formal power series with algebraic coefficients as the smallest integer k such that this series is the Hadamard product of k algebraic power series. In this paper, we obtain lower and upper bounds for the grade of hypergeometric series by comparing two different asymptotic expansions of their Taylor coefficients, one obtained from their definition and another one obtained when assuming that the grade has a certain value. In such expansions, Gamma values at rational points naturally appear and our results mostly depend on the Rohrlich–Lang Conjecture for polynomial relations in Gamma values. We also obtain unconditional and sharp results when we can apply Diophantine results such as the Wolfart–Wüstholz Theorem for Beta values.

1. Introduction

The main goal of this paper is to study the notion of grade of a power series introduced by Allouche and Mendès France in [1]. We recall that the Hadamard product

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$F(z) * G(z) \in \mathbb{C}[\![z]\!]$ of two formal power series $F(z) = \sum_{n\geq 0} f_n z^n \in \mathbb{C}[\![z]\!]$ and $G(z) = \sum_{n\geq 0} g_n z^n \in \mathbb{C}[\![z]\!]$ is defined by

$$F(z) * G(z) = \sum_{n\geq 0} f_n g_n z^n.$$

Definition 1. A formal power series $F(z) \in \overline{\mathbb{Q}}[\![z]\!]$ has finite grade over $\overline{\mathbb{Q}}$ if there exist $A_1(z), \ldots, A_m(z) \in \overline{\mathbb{Q}}[\![z]\!]$ algebraic over $\overline{\mathbb{Q}}(z)$ such that

$$F(z) = A_1(z) * \cdots * A_m(z). \tag{1.1}$$

If $F(z)$ has finite grade over $\overline{\mathbb{Q}}$ then the smallest integer $m \geq 1$ such that (1.1) holds for some $A_1(z), \ldots, A_m(z) \in \overline{\mathbb{Q}}[\![z]\!]$ algebraic over $\overline{\mathbb{Q}}(z)$ is denoted by $\mathrm{grade}_{\overline{\mathbb{Q}}}(F(z))$ and is called the grade of $F(z)$ over $\overline{\mathbb{Q}}$. If $F(z)$ does not have finite grade over $\overline{\mathbb{Q}}$, then we set $\mathrm{grade}_{\overline{\mathbb{Q}}}(F(z)) = \infty$.

From now on, by "algebraic function" or "algebraic series", we will mean a power series in $\overline{\mathbb{Q}}[\![z]\!]$ which is algebraic over $\overline{\mathbb{Q}}(z)$.

If $F(z) = \sum_{n\geq 0} f_n z^n \in \overline{\mathbb{Q}}[\![z]\!]$ has finite grade over $\overline{\mathbb{Q}}$ then it is a globally bounded G-function because algebraic functions are globally bounded G-functions (Abel, Eisenstein) and this property is preserved by Hadamard product (see [2, §VI.4, Corollary]). We recall that $F(z)$ is said to be globally bounded (terminology due to Christol [6]) if all the coefficients belong to some number field K and

- for every place ν of K, the ν-adic radius of convergence of $F(z)$ is non-zero;
- there exists some non-zero integer N such that the coefficients of F belong to $\mathcal{O}_K[1/N]$, where $\mathcal{O}_K$ is the ring of integers of K.

Moreover, $F(z)$ is a G-function if all the coefficients belong to some number field K and

- the maximum of the moduli of the conjugates of f_n grows at most geometrically with n;
- there exists a sequence of positive numbers $(d_n)_{n\geq 0}$ which grows at most geometrically with n such that $d_n a_n$ belongs to the ring of integers $\mathcal{O}_K$ of K;
- $F(z)$ satisfies some non-trivial homogeneous linear differential equation with coefficients in $K(z)$.

It follows for instance that $\log(1-z)$ and $(1-z)^\alpha$ with $\alpha \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$ do not have finite grade. It is very likely that there exist globally bounded G-functions with infinite grade; see Proposition 1.

It is in general very difficult to estimate the grade of a globally bounded G-function and Allouche–Mendès France raised a few questions in this respect, from which this