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Self-conjugate core partitions and modular forms

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A B S T R A C T

A recent paper by Hanusa and Nath states many conjectures in the study of self-conjugate core partitions. We prove all but two of these conjectures asymptotically by number-theoretic means. We also obtain exact formulas for the number of self-conjugate t-core partitions for "small" t via explicit computations with modular forms. For instance, self-conjugate 9-core partitions are related to counting points on elliptic curves over $\mathbb{Q}$ with conductor dividing 108, and self-conjugate 6-core partitions are related to the representations of integers congruent to 11 mod 24 by $3X^2 + 32Y^2 + 96Z^2$, a form with finitely many (conjecturally five) exceptional integers in this arithmetic progression, by an ineffective result of Duke–Schulze-Pillot.

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1. Introduction

Since the time of Young it has been known that partitions index the irreducible representations of the symmetric groups. Young and mathematicians of his time also knew that a partition could be encoded in a convenient way — via what is now known as a *Young diagram* — and that flipping this diagram about a natural diagonal amounted to tensoring the corresponding irreducible representation with the sign character. Hence it was deduced that the Young diagrams invariant under this flip corresponded to those irreducible representations that split upon restriction to the alternating subgroup.

Some time later, it was discovered by Frame–Robinson–Thrall [6] that the hook lengths of a Young diagram determine the dimension of the corresponding irreducible representation (over $\mathbb{C}$). It followed that the study of partitions with hook lengths indivisible by a given integer t — so-called *t-core partitions* — was connected to modular representation theory.

In this paper we study self-conjugate t-core partitions, asymptotically resolving all but two conjectures posed in the paper of Hanusa and Nath [10] on counting self-conjugate t-core partitions. In all but two cases the implied constants are effective, so in principle this reduces many of these conjectures to a finite amount of computation. The ineffective cases are due to the ineffectivity of a result of Duke–Schulze-Pillot [5] on integers represented by forms in a given spinor genus, which arises due to the Landau–Siegel phenomenon.

2. Preliminaries

Let $\lambda := \lambda_1 \leqslant \cdots \leqslant \lambda_k$ be a partition of n. For each box b in its associated Young diagram, one defines its hook length h_b by counting the number of boxes directly to its right or below it, including the box itself. The irreducible representations of the symmetric group on n letters, S_n, are in explicit bijection with the partitions of n. The *hook-length formula* states that the irreducible representation corresponding to λ has dimension

$$\dim \rho_\lambda = \frac{n!}{\prod h_b}, \tag{1}$$

the product taken over all the boxes in the Young diagram corresponding to λ.

The representations of S_n can be defined over $\mathbb{Z}$ (i.e., can be realized as maps $S_n \to \mathrm{GL}_d(\mathbb{Z})$), and so one may speak of reduction modulo a prime p. From modular