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A generated approximation of the gamma function related to Windschitl's formula

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A B S T R A C T

In this paper, based on Windschitl's formula, a generated approximation of the factorial function and some inequalities for the gamma function are established. Finally, for demonstrating the superiority of our new series over Windschitl's formula, Nemes' formula and three Mortici's formulas, some numerical computations are given.

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1. Introduction

It is well known that the big factorials are often manipulated in many situations in pure mathematics and other branches of science. The Stirling's formula is one of the most known formulas for approximation of the factorial function, it was known as

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n. \tag{1.1}$$

Up until now, many researchers made great efforts in the area of establishing more precise inequalities and more accurate approximations for the factorial function and its extension gamma function, and had a lot of inspiring results. For example, the Stirling series [1],

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51\,840n^3} - \frac{571}{2\,488\,320n^4} + \cdots\right). \tag{1.2}$$

The more exact formula without simple shape is Windschitl's formula [12],

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(n \sinh\frac{1}{n}\right)^{n/2}. \tag{1.3}$$

Nemes' formula [11],

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n^2 - \frac{1}{10}}\right)^n, \tag{1.4}$$

which has the same number of exact digits as (1.3) but is much simpler.

In addition, recently, some authors paid attention to giving increasingly better approximations for the gamma function using continued fractions. For example, on the one hand, using own method, Mortici [9] rediscovered Stieltjes' continued fraction

$$\Gamma(x+1) \approx \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\frac{a_0}{x + \frac{a_1}{x + \frac{a_2}{x + \cdots}}}\right), \tag{1.5}$$

where

$$a_0 = 1/12, \qquad a_1 = 1/30, \qquad a_2 = 53/210 \quad \text{etc.}$$

On the other hand, Mortici [10] provided a new continued fraction approximation starting from the Nemes' formula (1.4) as follows,

$$\Gamma(x+1) \approx \sqrt{2\pi x}\, e^{-x}\left(x + \frac{1}{12x - \frac{1}{10x + \frac{a}{x + \frac{b}{x + \frac{c}{x + \frac{d}{x + \ddots}}}}}}\right)^x, \tag{1.6}$$