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New parity results for broken 11-diamond partitions

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The notation of broken k-diamond partitions was introduced in 2007 by Andrews and Paule. For a fixed positive integer k, let $\Delta_k(n)$ denote the number of broken k-diamond partitions of n. Recently, Radu and Sellers established numerous congruence properties for $(2k+1)$-cores by using the theory of modular forms, where $k = 2, 3, 5, 6, 8, 9, 11$. Employing their congruences for $(2k+1)$-cores, Radu and Sellers obtained a number of nice parity results for $\Delta_k(n)$. In particular, they proved that for $n \geqslant 0$, $\Delta_{11}(46n + r) \equiv 0 \pmod 2$, where $r \in \{11, 15, 21, 23, 29, 31, 35, 39, 41, 43, 45\}$. In this paper, we derive several new infinite families of congruences modulo 2 for $\Delta_{11}(n)$ by using an identity given by Chan and Toh, and the p-dissection of Ramanujan's theta function f_1 due to Cui and Gu. For example, we prove that for $n \geqslant 0$ and $k, \alpha \geqslant 1$, $\Delta_{11}(2^{3\alpha-2} \times 23^k n + 2^{3\alpha-2} s \times 23^{k-1} + 1) \equiv 0 \pmod 2$, where $s \in \{5, 7, 10, 11, 14, 15, 17, 19, 20, 21, 22\}$. This generalizes the parity results for $\Delta_{11}(n)$ discovered by Radu and Sellers.

1. Introduction

In 2007, Andrews and Paule introduced a new class of combinatorial objects called broken k-diamond partitions. For a fixed positive integer k, let $\Delta_k(n)$ denote the number of broken k-diamond partitions of n. Andrews and Paule [1] proved that

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$$\sum_{n=0}^{\infty}\Delta_k(n)q^n = \frac{f_2 f_{2k+1}}{f_1^3 f_{4k+2}}, \tag{1.1}$$

where here and throughout this paper, for any positive integer k, f_k is defined by

$$f_k = \prod_{n=1}^{\infty}(1-q^{nk}). \tag{1.2}$$

Employing generating function manipulations, Andrews and Paule [1] proved that for $n \geqslant 0$,

$$\Delta_1(2n+1) \equiv 0 \pmod{3}. \tag{1.3}$$

They also presented several conjectures modulo 2 for $\Delta_k(n)$. Since then, numerous mathematicians have considered partition congruences satisfied by $\Delta_k(n)$ for small values of k; see, for example, Chan [3], Fu [6], Hirschhorn and Sellers [7], Jameson [8], Mortenson [9], Paule and Radu [10], Radu and Sellers [11–13] and Xiong [14]. Recently, Radu and Sellers [11] have given numerous beautiful congruence properties for broken k-diamond partitions and $(2k+1)$-core partitions by using the theory of modular forms. In particular, they proved that for $n \geqslant 0$,

$$\Delta_{11}(46n+r) \equiv 0 \pmod{2}, \tag{1.4}$$

where $r \in \{11, 15, 21, 23, 29, 31, 35, 39, 41, 43, 45\}$.

The objective of this paper is to establish several infinite families of congruences modulo 2 for the number of broken 11-diamond partitions by employing an identity due to Chan and Toh [4], and the p-dissection formula of Ramanujan's theta function f_1 given by Cui and Gu [5]. In particular, our results generalize the parity results for the number of broken 11-diamond partitions proved by Radu and Sellers [11].

In order to state the main results, we introduce the Legendre symbol. Let $p \geqslant 3$ be a prime. The Legendre symbol $(\frac{a}{p})$ is defined by

$$\left(\frac{a}{p}\right) := \begin{cases} 1, & \text{if } a \text{ is a quadratic residue modulo } p \text{ and } p \nmid a,\\ -1, & \text{if } a \text{ is a quadratic nonresidue modulo } p,\\ 0, & \text{if } p \mid a. \end{cases} \tag{1.5}$$

The main results of this paper can be stated as follows.

Theorem 1.1. *For $n \geqslant 0$, $k, \alpha \geqslant 1$ and any prime p such that $(\frac{-23}{p}) = -1$,*

$$\Delta_{11}\big(2^{3\alpha+1}n + 2^{3\alpha} + 1\big) \equiv 0 \pmod{2}, \tag{1.6}$$

$$\Delta_{11}\big(2^{3\alpha-2}\times 23^k n + 2^{3\alpha-2}\times 23^{k-1}s + 1\big) \equiv 0 \pmod{2}, \tag{1.7}$$

$$\Delta_{11}\big(2\times 23^k p^{2\alpha-2}n + 2\times 23^{k-1}p^{2\alpha-2}s + 1\big) \equiv 0 \pmod{2}, \tag{1.8}$$