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Divisibility by 2 of Stirling numbers of the second kind and their differences

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ABSTRACT

Let n, k, a and c be positive integers and b be a nonnegative integer. Let $\nu_2(k)$ and $s_2(k)$ be the 2-adic valuation of k and the sum of binary digits of k, respectively. Let $S(n,k)$ be the Stirling number of the second kind. It is shown that $\nu_2(S(c2^n, b2^{n+1}+a)) \geqslant s_2(a)-1$, where $0<a<2^{n+1}$ and $2 \nmid c$. Furthermore, one gets that $\nu_2(S(c2^n, (c-1)2^n+a)) = s_2(a)-1$, where $n \geqslant 2$, $1 \leqslant a \leqslant 2^n$ and $2 \nmid c$. Finally, it is proved that if $3 \leqslant k \leqslant 2^n$ and k is not a power of 2 minus 1, then $\nu_2(S(a2^n,k)-S(b2^n,k)) = n+\nu_2(a-b)-\lceil \log_2 k \rceil + s_2(k)+\delta(k)$, where $\delta(4)=2$, $\delta(k)=1$ if $k>4$ is a power of 2, and $\delta(k)=0$ otherwise. This confirms a conjecture of Lengyel raised in 2009 except when k is a power of 2 minus 1.

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1. Introduction and the statements of main results

The Stirling number of the second kind $S(n,k)$ is defined for $n \in \mathbb{N}$ and positive integer $k \leqslant n$ as the number of ways to partition a set of n elements into exactly k non-empty subsets. It satisfies the recurrence relation

$$S(n,k) = S(n-1,k-1) + kS(n-1,k),$$

with initial condition $S(0,0) = 1$ and $S(n,0) = 0$ for $n > 0$. There is also an explicit formula in terms of binomial coefficients given by

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n. \tag{1}$$

Divisibility properties of Stirling numbers have been studied from a number of different perspectives. It is known that for each fixed k, the sequence $\{S(n,k),\ n \geqslant k\}$ is periodic modulo prime powers. The length of this period has been studied by Carlitz [5] and Kwong [16]. Chan and Manna [6] characterized $S(n,k)$ modulo prime powers in terms of binomial coefficients. In fact, they gave explicit formulas for $S(n,k)$ modulo 4, then for $S(n,a2^m)$ modulo 2^m, where $m \geqslant 3$, $a > 0$ and $n \geqslant a2^m + 1$, and finally for $S(n,ap^m)$ modulo p^m with p being an odd prime.

Divisibility properties of integer sequences are often expressed in terms of p-adic valuations. Given a prime p and a positive integer m, there exist unique integers a and n, with $p \nmid a$ and $n \geqslant 0$, such that $m = ap^n$. The number n is called the *p-adic valuation* of m, denoted by $n = \nu_p(m)$. The numbers $\min\{\nu_p(k!S(n,k)):\ m \leqslant k \leqslant n\}$ are important in algebraic topology, see, for example, [3,8,10–12,20,21]. Some work on evaluating $\nu_p(k!S(n,k))$ has appeared in above papers as well as in [7,9,24]. Amdeberhan, Manna and Moll [2] investigated the 2-adic valuations of Stirling numbers of the second kind and computed $\nu_2(S(n,k))$ for $k \leqslant 4$. They also raised an interesting conjecture on the congruence classes of $S(n,k)$, modulo powers of 2. Recently, Bennett and Mosteig [4] used computational methods to justify this conjecture if $k \leqslant 20$. But this conjecture is still kept open if $k \geqslant 21$.

This paper deals with the 2-adic valuations of the Stirling numbers of the second kind. Lengyel [17] studied the 2-adic valuations of $S(n,k)$ and conjectured, proved by Wannemacker [23], $\nu_2(S(2^n,k)) = s_2(k) - 1$, where $s_2(k)$ means the base 2 digital sum of k. Using Wannemacker's result, Hong, Zhao and Zhao [13] proved that $\nu_2(S(2^n+1, k+1)) = s_2(k) - 1$, which confirmed another conjecture of Amdeberhan, Manna and Moll [2]. Lengyel [18,19] showed that if $1 \leqslant k \leqslant 2^n$, then $\nu_2(S(c2^n,k)) = s_2(k) - 1$ for any positive integer c. Meanwhile, Lengyel [18] proved that $\nu_2(S(c2^n,k)) \geqslant s_2(k) - 1$ if $c \geqslant 1$ is an odd integer and $1 \leqslant k \leqslant 2^{n+1}$. Actually, a more general result is true. That is, one has