Journal of Number Theory 144 (2014) 15–24

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

A curious congruence modulo prime powers ☆

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ARTICLE INFO

Article history:
Received 14 February 2014
Received in revised form 11 April 2014
Accepted 11 April 2014
Available online 4 June 2014
Communicated by David Goss

MSC:
11A07
11A41

Keywords:
Congruences
Bernoulli numbers
Modulo prime powers

ABSTRACT

Zhao established a curious congruence, i.e., for any prime $p \geq 3$,

$$\sum_{\substack{i+j+k=p\\ i,j,k>0}} \frac{1}{ijk} \equiv -2B_{p-3} \pmod{p}.$$

In this note we prove that for any prime $p \geq 3$ and positive integer r,

$$\sum_{\substack{i+j+k=p^r\\ i,j,k\in\mathcal{P}_p}} \frac{1}{ijk} \equiv -2p^{r-1}B_{p-3} \pmod{p^r},$$

where $\mathcal{P}_n$ denotes the set of positive integers which are prime to n.

☆ The project supported by the National Natural Science Foundation of China 11351002.

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http://dx.doi.org/10.1016/j.jnt.2014.04.004
0022-314X/

1. Introduction

Jianqiang Zhao [8] first found a curious congruence

$$\sum_{\substack{i+j+k=p\\ i,j,k>0}} \frac{1}{ijk} \equiv -2B_{p-3} \pmod p, \tag{1.1}$$

where $p \geq 3$ is a prime, and B_n is the n-th Bernoulli number.

Chungang Ji [3] gave a simple proof for this congruence and later Xia Zhou and Tianxin Cai [9] gave a generalized form of (1.1)

$$\sum_{\substack{l_1+l_2+\cdots+l_n=p,\\ l_1,l_2,\cdots,l_n>0}} \frac{1}{l_1 l_2 \cdots l_n} \equiv \begin{cases} -(n-1)!B_{p-n} \pmod p & \text{if } 2 \nmid n, \\ -\frac{n}{2(n+1)} n! B_{p-n-1} p \pmod{p^2} & \text{if } 2|n \end{cases}$$

where $p \geq 5$ is a prime and $n \leq p-2$ is a positive integer.

Meanwhile, Binzhou Xia and Tianxin Cai [7] generalized (1.1) to

$$\sum_{\substack{i+j+k=p\\ i,j,k>0}} \frac{1}{ijk} \equiv \frac{12B_{p-3}}{p-3} - \frac{3B_{2p-4}}{p-2} \pmod{p^2},$$

where $p > 5$ is a prime.

In this paper, we obtain the following theorems.

Theorem 1. *Let $p \geq 3$ be a prime and r a positive integer, then*

$$\sum_{\substack{i+j+k=p^r\\ i,j,k\in\mathcal{P}_p}} \frac{1}{ijk} \equiv -2p^{r-1}B_{p-3} \pmod{p^r},$$

where $\mathcal{P}_n$ denotes the set of positive integers which are prime to n.

In particular, if $r = 1$, Theorem 1 becomes (1.1).

Theorem 2. *Let n be a positive integer and $p \geq 3$ a prime, if $p^r|n$, $r \geq 1$ is an integer, then*

$$\sum_{\substack{i+j+k=n\\ i,j,k\in\mathcal{P}_p}} \frac{1}{ijk} \equiv -\frac{2n}{p}B_{p-3} \pmod{p^r}.$$

In particular, if $n = p^r$, Theorem 2 becomes Theorem 1.