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Newman's conjecture in various settings ☆

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A B S T R A C T

Text. De Bruijn and Newman introduced a deformation of the Riemann zeta function $\zeta(s)$, and found a real constant Λ which encodes the movement of the zeros of $\zeta(s)$ under the deformation. The Riemann hypothesis is equivalent to $\Lambda \leq 0$. Newman conjectured $\Lambda \geq 0$, remarking "the new conjecture is a quantitative version of the dictum that the Riemann hypothesis, if true, is only barely so." Previous work could only handle $\zeta(s)$ and quadratic Dirichlet L-functions, obtaining lower bounds very close to zero ($-1.14541 \cdot 10^{-11}$ for $\zeta(s)$ and $-1.17 \cdot 10^{-7}$ for quadratic Dirichlet L-functions). We generalize to automorphic L-functions and function field L-functions, and explore the limit of these techniques. If $\mathcal{D} \in \mathbb{Z}[T]$ is a square-free polynomial of degree 3 and D_p the polynomial in $\mathbb{F}_p[T]$ obtained by reducing $\mathcal{D}$ modulo p, we prove the Newman constant Λ_{D_p} equals $\log \frac{|a_p(\mathcal{D})|}{2\sqrt{p}}$; by Sato–Tate (if the curve is non-CM) there exists a sequence

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of primes such that $\lim_{n\to\infty} \Lambda_{D_{p^n}} = 0$. We end by discussing connections with random matrix theory.

Video. For a video summary of this paper, please visit http://youtu.be/8A1XZtSkp_Q. This author video is a recording of a talk given by Alan Chang at CANT on May 28, 2014.

Contents

1. Introduction

1.1. Newman's conjecture for the Riemann zeta function

Let

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) \tag{1.1}$$

be the completed Riemann zeta function, and let

$$\Xi(x) = \xi\left(\frac{1}{2} + ix\right). \tag{1.2}$$

Because of the functional equation $\xi(s) = \xi(1-s)$, we know that $x \in \mathbb{R}$ implies $\Xi(x) \in \mathbb{R}$. In general, we allow x to be complex.