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The least common multiple of random sets of positive integers

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We study the typical behavior of the least common multiple of the elements of a random subset $A \subset \{1, \dots, n\}$. For example we prove that $\operatorname{lcm}\{a\colon a \in A\} = 2^{n(1+o(1))}$ for almost all subsets $A \subset \{1, \dots, n\}$.

1. Introduction

The function $\psi(n) = \log \operatorname{lcm}\{m\colon 1 \le m \le n\}$ was introduced by Chebyshev in his study on the distribution of the prime numbers. It is a well known fact that the

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asymptotic relation $\psi(n) \sim n$ is equivalent to the Prime Number Theorem, which was proved independently by J. Hadamard and C.J. de la Vallée Poussin.

In the present paper, instead of considering the whole set $\{1, \dots, n\}$, we study the typical behavior of the quantity $\psi(A) := \log \operatorname{lcm}\{a\colon a \in A\}$ for a random set A in $\{1, \dots, n\}$ when $n \to \infty$. We define $\psi(\emptyset) = 0$. We consider two natural models.

In the first one, denoted by $B(n, \delta)$, each element in A is chosen independently at random in $\{1, \dots, n\}$ with probability $\delta = \delta(n)$, typically a function of n.

Theorem 1.1. *If $\delta = \delta(n) < 1$ and $\delta n \to \infty$ then*

$$\psi(A) \sim n\frac{\delta \log(\delta^{-1})}{1 - \delta}$$

asymptotically almost surely in $B(n, \delta)$ when $n \to \infty$.

The case $\delta = 1$ corresponds to the classical Chebyshev function and its asymptotic estimate appears as the limiting case, as δ tends to 1, in Theorem 1.1, since $\lim_{\delta \to 1} \frac{\delta \log(\delta^{-1})}{1-\delta} = 1$.

When $\delta = 1/2$ all the subsets $A \subset \{1, \dots, n\}$ are chosen with the same probability and Theorem 1.1 gives the following result.

Corollary 1.1. *For almost all sets $A \subset \{1, \dots, n\}$ we have that*

$$\operatorname{lcm}\{a\colon a \in A\} = 2^{n(1+o(1))}.$$

For a given positive integer $k = k(n)$, again typically a function of n, we consider the second model, where each subset of k elements is chosen uniformly at random among all sets of size k in $\{1, \dots, n\}$. We denote this model by $S(n, k)$.

When $\delta = k/n$ the heuristic suggests that both models are quite similar. Indeed, this is the strategy we follow to prove Theorem 1.2.

Theorem 1.2. *For $k = k(n) < n$ and $k \to \infty$ we have*

$$\psi(A) = k\frac{\log(n/k)}{1 - k/n}\bigl(1 + O\bigl(e^{-C\sqrt{\log k}}\bigr)\bigr)$$

almost surely in $S(n, k)$ when $n \to \infty$ for some positive constant C.

The case $k = n$, which corresponds to Chebyshev's function, is also obtained as a limiting case in Theorem 1.2 in the sense that $\lim_{k/n \to 1} \frac{\log(n/k)}{1-k/n} = 1$.

This work has been motivated by a result of the first author about the asymptotic behavior of $\psi(A)$ when $A = A_{q,n} := \{q(m)\colon 1 \le q(m) \le n\}$ for a quadratic polynomial $q(x) \in \mathbb{Z}[x]$. We wondered if that behavior was typical among the sets $A \subset \{1, \dots, n\}$ of similar size. We analyze this issue in the last section.