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Supercharacters, exponential sums, and the uncertainty principle ☆

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A B S T R A C T

The theory of supercharacters, which generalizes classical character theory, was recently introduced by P. Diaconis and I.M. Isaacs, building upon earlier work of C. André. We study supercharacter theories on $(\mathbb{Z}/n\mathbb{Z})^d$ induced by the actions of certain matrix groups, demonstrating that a variety of

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Kloosterman sum
Symmetric group
Uncertainty principle

exponential sums of interest in number theory (e.g., Gauss, Ramanujan, Heilbronn, and Kloosterman sums) arise in this manner. We develop a generalization of the discrete Fourier transform, in which supercharacters play the role of the Fourier exponential basis. We provide a corresponding uncertainty principle and compute the associated constants in several cases.

1. Introduction

The theory of *supercharacters*, of which classical character theory is a special case, was recently introduced by P. Diaconis and I.M. Isaacs in 2008 [6], generalizing the *basic characters* studied by C. André [1–3]. We are interested here in supercharacter theories on the group $(\mathbb{Z}/n\mathbb{Z})^d$ induced by the action of certain subgroups Γ of the group $GL_d(\mathbb{Z}/n\mathbb{Z})$ of invertible $d \times d$ matrices over $\mathbb{Z}/n\mathbb{Z}$. In particular, we demonstrate that a variety of exponential sums which are of interest in number theory arise as supercharacter values. Among the examples we discuss are Gauss, Ramanujan, Heilbronn, and Kloosterman sums. Moreover, we also introduce a class of exponential sums induced by the natural action of the symmetric group S_d on $(\mathbb{Z}/n\mathbb{Z})^d$ that yields some visually striking patterns.

In addition to showing that the machinery of supercharacter theory can be used to generate identities for certain exponential sums, we also develop a generalization of the discrete Fourier transform in which supercharacters play the role of the Fourier exponential basis. For the resulting *super-Fourier transform*, we derive a supercharacter analogue of the uncertainty principle. We also describe the algebra of all operators that are diagonalized by our transform. Some of this is reminiscent of the theory of Fourier transforms of characteristic functions of orbits in Lie algebras over finite fields [15, Lem. 3.1.10], [14, Lem. 4.2], [19].

Although it is possible to derive some of our results by considering the classical character theory of the semidirect product $(\mathbb{Z}/n\mathbb{Z})^d \rtimes \Gamma$, the supercharacter approach is cleaner and more natural. The character tables produced via the classical approach are typically large and unwieldy, containing many entries that are irrelevant to the study of the particular exponential sum being considered. This is a reflection of the fact that $(\mathbb{Z}/n\mathbb{Z})^d \rtimes \Gamma$ is generally nonabelian and possesses a large number of conjugacy classes. On the other hand, our supercharacter tables are smaller and simpler than their classical counterparts. Indeed, the supercharacter approach takes place entirely inside the original abelian group $(\mathbb{Z}/n\mathbb{Z})^d$, which possesses only a few superclasses.