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Blueprints — towards absolute arithmetic?

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ABSTRACT

One of the driving motivations for $\mathbb{F}_1$-geometry is the hope to translate Weil's proof of the Riemann hypothesis from positive characteristics to number fields. The spectrum of $\mathbb{Z}$ should find an interpretation as a curve over $\mathbb{F}_1$, together with a completion $\overline{\operatorname{Spec}\mathbb{Z}}$. Intersection theory for divisors on the arithmetic surface $\overline{\operatorname{Spec}\mathbb{Z}} \times \overline{\operatorname{Spec}\mathbb{Z}}$ should allow us to mimic Weil's proof. It is possible to define $\overline{\operatorname{Spec}\mathbb{Z}}$ as a locally blueprinted space, which shares certain properties with its analog in positive characteristic. In particular, the arithmetic surface $\overline{\operatorname{Spec}\mathbb{Z}} \times \overline{\operatorname{Spec}\mathbb{Z}}$ is two-dimensional. We describe the local factors (including the Γ-factor) of the Riemann zeta function as integrals over the space of ideals of the stalks of the structure sheaf of $\overline{\operatorname{Spec}\mathbb{Z}}$. A comparison of line bundles on $\overline{\operatorname{Spec}\mathbb{Z}}$ with Arakelov divisor exhibits a second integral formula for the Riemann zeta function. We conclude this note with some remarks on étale cohomology for $\overline{\operatorname{Spec}\mathbb{Z}}$.

0. Introduction

For a not yet systematically understood reason, many arithmetic laws have (conjectural) analogues for function fields and number fields. While in the function field case, these laws often have a conceptual explanation by means of a geometric interpretation,

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methods from algebraic geometry break down in the number field case. The mathematical area of $\mathbb{F}_1$-geometry can be understood as a program to develop a geometric language that allows us to transfer the geometric methods from function fields to number fields.

A central problem of this kind, which lacks a proof in the number field case, is the Riemann hypothesis. The Riemann zeta function is defined as the Riemann sum $\zeta(s) = \sum_{n\geq 1} n^{-s}$, or, equivalently, as the Euler product $\prod_{p \text{ prime}}(1-p^{-s})^{-1}$ (these expressions converge for $\mathrm{Re}(s) > 1$, but can be continued to meromorphic functions on $\mathbb{C}$). A more symmetric expression with respect to its functional equation is given by the *completed zeta function*

$$\zeta^*(s) = \underbrace{\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)}_{\zeta_\infty(s)} \cdot \prod_{p \text{ prime}} \underbrace{\frac{1}{1-p^{-s}}}_{\zeta_p(s)} \quad \bigl(\text{for } \mathrm{Re}(s) > 1\bigr)$$

where we call the factor $\zeta_p(s)$ the *local zeta factor at p* for $p \leq \infty$. The meromorphic continuation of $\zeta^*(s)$ to $\mathbb{C}$ satisfies $\zeta^*(s) = \zeta^*(1-s)$. The fundamental conjecture is the

Riemann hypothesis: If $\zeta^*(s) = 0$, then $\mathrm{Re}(s) = 1/2$.

The analogous statement for the function field of a curve X over a finite field has been proven by André Weil more than seventy years ago. Weil's proof uses intersection theory for the self-product $X \times X$ and the Lefschetz fix-point formula for the absolute Frobenius action on X resp. $X \times X$. There were several attempts to translate the geometric methods of this proof into arithmetic arguments that would apply for number fields as well, but only with partial success so far.

A different approach, primarily due to Grothendieck, is the development of a theory of (mixed) motives, but such a theory relies on the solution of some fundamental problems. In particular, Deninger formulates in [2] a conjectural formalism for a *big site* $\mathscr{T}$ of motives that should contain a compactification $\overline{\operatorname{Spec}\mathbb{Z}} = \operatorname{Spec}\mathbb{Z} \cup \{\infty\}$ of the arithmetic line, and he conjectured that the formula

$$\zeta^*(s) = \frac{\det_\infty(\frac{1}{2\pi}(s-\Theta)|H^1(\overline{\operatorname{Spec}\mathbb{Z}}, \mathscr{O}_{\mathscr{T}}))}{\det_\infty(\frac{1}{2\pi}(s-\Theta)|H^0(\overline{\operatorname{Spec}\mathbb{Z}}, \mathscr{O}_{\mathscr{T}})) \cdot \det_\infty(\frac{1}{2\pi}(s-\Theta)|H^2(\overline{\operatorname{Spec}\mathbb{Z}}, \mathscr{O}_{\mathscr{T}}))}$$

holds true where $\det_\infty$ denotes the regularized determinant, Θ is an endofunctor on $\mathscr{T}$ and $H^i(-, \mathscr{O}_{\mathscr{T}})$ is a certain proposed cohomology. This description combines with Kurokawa's work on multiple zeta functions [7] to the hope that there are motives h^0 (*"the absolute point"*), h^1 and h^2 (*"the absolute Tate motive"*) with zeta functions

$$\zeta_{h^w}(s) = \det{}_\infty\left(\frac{1}{2\pi}(s-\Theta)\middle|H^w(\overline{\operatorname{Spec}\mathbb{Z}}, \mathscr{O}_{\mathscr{T}})\right)$$

The viewpoint of $\mathbb{F}_1$-geometry is that $\overline{\operatorname{Spec}\mathbb{Z}}$ should be a curve over the elusive field $\mathbb{F}_1$ with one element. A good theory of geometry over $\mathbb{F}_1$ might eventually allow us to