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Biquadratic Pólya fields with only one quadratic Pólya subfield

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ABSTRACT

A number field is called a Pólya field if the module of integer-valued polynomials over its integers has a regular basis. In this paper, we answer two open questions raised in [4] on biquadratic Pólya fields using some results of [8] and [2] on Galois cohomology of number fields. In particular, we find infinitely many biquadratic Pólya fields with only one quadratic Pólya subfield.

1. Introduction

Let K be an algebraic number field and denote by $\mathcal{O}_K$ its ring of integers. Consider the ring of integer-valued polynomials on $\mathcal{O}_K$:

$$\mathrm{Int}(\mathcal{O}_K) = \{ f \in K[X] \mid f(\mathcal{O}_K) \subseteq \mathcal{O}_K \}.$$

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Definition 1.1. (See [6,8].) A number field K is said to be a Pólya field if the $\mathcal{O}_K$-module $\mathrm{Int}(\mathcal{O}_K)$ has a regular basis, i.e. a basis (f_n) such that for each $n \in \mathbb{N} \cup \{0\}$, $\deg(f_n) = n$.

For each $n \in \mathbb{N}$, the leading coefficients of the polynomials in $\mathrm{Int}(\mathcal{O}_K)$ of degree n together with zero forms a fractional ideal of $\mathcal{O}_K$, denoted by $\mathfrak{J}_n(K)$ [3]. It is easy to see that K is a Pólya field if and only if the ideals $\mathfrak{J}_n(K)$ are principal [3].

Definition 1.2. (See [3].) The Pólya–Ostrowski group (or shortly, the Pólya group) of K is the subgroup $Po(K)$ of $Cl(K)$ (the class group of $\mathcal{O}_K$) generated by the classes of the ideals $\mathfrak{J}_n(K)$.

Now for each $q \geq 2$, let $\Pi_q(K)$ be the product of all maximal ideals of $\mathcal{O}_K$ with norm q. If q is not the norm of an ideal, then define $\Pi_q(K)$ to be $\mathcal{O}_K$.

Ostrowski [5] proved that $Po(K)$ is generated by the classes of the ideals $\Pi_q(K)$ in $Cl(K)$ (see [3]). Therefore, if K is a Galois extension of $\mathbb{Q}$, then K is a Pólya field if and only if $\Pi_q(K)$ is principal for powers of ramified primes as q. Zantema [8] completely characterized quadratic Pólya fields (see [3]):

Proposition 1.3. *A quadratic field $\mathbb{Q}[\sqrt{d}]$ for square-free d is a Pólya field if and only if d is one of the following:*

i) $d = -1$, $d = 2$, $d = -2$,
ii) $d = -p$, *where p is prime and* $p \equiv 3 \pmod 4$,
iii) $d = p$, *where p is an odd prime,*
iv) $d = 2p$, *where p is prime and*
  - *either* $p \equiv 3 \pmod 4$
  - *or* $p \equiv 1 \pmod 4$ *and the fundamental unit has norm* $+1$,
v) $d = pq$, *where p and q are primes and*
  - *either* $p, q \equiv 3 \pmod 4$
  - *or* $p, q \equiv 1 \pmod 4$ *and the fundamental unit has norm* $+1$.

Theorem 1.4. *(See [8].) Let K_1 and K_2 be finite Galois extensions of $\mathbb{Q}$, $L = K_1.K_2$, $K = K_1 \cap K_2$. For a prime p of K, let $e_i(p)$ be the ramification index of p in K_i, $i = 1, 2$. Then:*

a) *If $\gcd(e_1(p), e_2(p)) = 1$ for all primes p, and K_1 and K_2 are Pólya fields, then L is a Pólya field.*
b) *If either $[K_1 : K]$, $[K_2 : K]$, $[K : \mathbb{Q}]$ are pairwise relatively prime, or K is Pólya and $\gcd([K_1 : K], [K_2 : K]) = 1$, then K_1 and K_2 are Pólya fields if L is a Pólya field.*

After quadratic fields, the case of biquadratic number fields seems a natural choice. The following theorem is proved in [4]: