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Limiting value of higher Mahler measure

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We consider the k-higher Mahler measure $m_k(P)$ of a Laurent polynomial P as the integral of $\log^k |P|$ over the complex unit circle. In this paper we derive an explicit formula for the value of $|m_k(P)|/k!$ as $k \to \infty$.

1. Introduction

For a non-zero Laurent polynomial $P(z) \in \mathbb{C}[z, z^{-1}]$, the k-higher Mahler measure of P is defined [4] as

$$m_k(P) = \int_0^1 \log^k \left| P\left(e^{2\pi it}\right) \right| \mathrm{d}t.$$

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For $k = 1$ this coincides with the classical (log) Mahler measure defined as

$$m(P) = \log|a| + \sum_{j=1}^{n} \log\bigl(\max\{1, |r_j|\}\bigr), \quad \text{for } P(z) = a\prod_{j=1}^{n}(z - r_j),$$

since by Jensen's formula $m(P) = m_1(P)$ [3].

Though classical Mahler measure was studied extensively, higher Mahler measure was introduced and studied very recently by Kurokawa, Lalín and Ochiai [4] and Akatsuka [1]. It is very difficult to evaluate k-higher Mahler measure for polynomials except few specific examples shown in [1] and [4], but it is relatively easy to find their limiting values.

In [5] Lalín and Sinha answered Lehmer's question [3] for higher Mahler measure by finding non-trivial lower bounds for m_k on $\mathbb{Z}[z]$ for $k \geq 2$.

In [2] it has been shown using Akatsuka's zeta function of [4] that for $|a| = 1$, $|m_k(z + a)|/k! \to 1/\pi$ as $k \to \infty$. In this paper we generalize this result by computing the same limit for an arbitrary Laurent polynomial $P(z) \in \mathbb{C}[z, z^{-1}]$ using a different technique.

Theorem 1.1. *Let $P(z) \in \mathbb{C}[z, z^{-1}]$ be a Laurent polynomial, possibly with repeated roots. Let $z_1, \ldots, z_n$ be the distinct roots of P. Then*

$$\lim_{k\to\infty} \frac{|m_k(P)|}{k!} = \frac{1}{\pi}\sum_{z_j \in S^1} \frac{1}{|P'(z_j)|},$$

where S^1 is the complex unit circle $|z| = 1$, and the right-hand side is taken as ∞ if $P'(z_j) = 0$ for some $z_j \in S^1$, i.e., if P has a repeated root on S^1.

2. Proof of the theorem

We first prove several lemmas which essentially show that the integrand may be linearly approximated near the roots of P on S^1.

Lemma 2.1. *Let $P(z) \in \mathbb{C}[z, z^{-1}]$ be a Laurent polynomial and $A \subseteq [0, 1]$ be a closed set such that $P(e^{2\pi it}) \neq 0$ for all $t \in A$. Then*

$$\lim_{k\to\infty} \frac{1}{k!} \int_A \log^k\bigl|P(e^{2\pi it})\bigr|\, dt = 0.$$

Proof. Since A is closed, due to the periodicity of $e^{2\pi it}$ and continuity of $P(e^{2\pi it})$ there exist constants b and B such that $0 < b \leq |P(e^{2\pi it})| \leq B$ on A. Then for each positive integer k, $(\log^k |P(e^{2\pi it})|)/k!$ is bounded between $(\log^k b)/k!$ and $(\log^k B)/k!$, and therefore $(1/k!) \int_A \log^k |P(e^{2\pi it})|\, dt$ is bounded between $(\mu A \log^k b)/k!$ and $(\mu A \log^k B)/k!$, where μA is the Lebesgue measure of A. The result follows by letting k tend to infinity. $\square$