Journal of Number Theory 143 (2014) 363–384

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Density of values of linear maps on quadratic surfaces

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ARTICLE INFO

Article history:
Received 24 January 2013
Received in revised form 30 January 2014
Accepted 28 April 2014
Available online 10 June 2014
Communicated by Ph. Michel

Keywords:
Quadratic forms
Linear maps
Density of values
Ratner's orbit closure theorem

ABSTRACT

In this paper we investigate the density properties of the set of values of a linear map at integer points on a quadratic surface. In particular we show that this set is dense in the range of the linear map subject to certain algebraic conditions on the linear map and the quadratic form that defines the surface. The proof uses Ratner's theorem on orbit closures of unipotent subgroups acting on homogeneous spaces.

1. Introduction

We are motivated by the following general problem.

Problem 1.1. If X is some rational surface in $\mathbb{R}^d$ and $P : X \to \mathbb{R}^s$ is a polynomial map, then what can one say about the distribution of the set $\{P(x) : x \in X \cap \mathbb{Z}^d\}$ in $\mathbb{R}^s$?

One expects to be able to answer Problem 1.1 by showing that the set $\{P(x) : x \in X \cap \mathbb{Z}^d\}$ is dense in $\mathbb{R}^s$ under certain dimension and rationality conditions imposed

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http://dx.doi.org/10.1016/j.jnt.2014.04.020

on P. In full generality Problem 1.1 is unapproachable via available techniques and what is known is limited to special cases. For instance, when $X = \mathbb{R}^d$ Problem 1.1 has been considered for P a linear or quadratic map, or combinations of both. The case when P is linear is classical, and is treated by Theorem 1 on page 64 of [2]. When P is quadratic, Problem 1.1 is known as the Oppenheim conjecture, density was first established by the work of G. Margulis in [10] and subsequently refined by S.G. Dani and G. Margulis in [4]. Moreover, in this situation qualitative results have also been established; initially by S.G. Dani and G. Margulis in [6] and later by A. Eskin, S. Mozes and G. Margulis in [7]. For P, a pair, consisting of a quadratic and linear form, Problem 1.1 has been considered by A. Gorodnik in [8]. The case when P consists of a system of many linear forms and a quadratic form has been considered by S.G. Dani in [3]. A. Gorodnik also considered the case when P consists of a system of quadratic forms in [9]. To the authors knowledge the case when $X \neq \mathbb{R}^d$ has not been considered. The main result of this paper deals with a case of Problem 1.1 when X is a quadratic surface and is stated below. Recall that any quadratic form Q on $\mathbb{R}^d$ can be uniquely associated with a symmetric matrix in $\mathrm{Mat}_d(\mathbb{R})$. We will write $\mathrm{rank}(Q)$ to mean the rank of the symmetric matrix associated to Q.

Theorem 1.2. *Suppose Q is a quadratic form on $\mathbb{R}^d$ such that Q is non-degenerate, indefinite with rational coefficients. For $a \in \mathbb{Q}$ define $X^a_{\mathbb{R}} = \{x \in \mathbb{R}^d : Q(x) = a\}$ and $X^a_{\mathbb{Z}} = X^a_{\mathbb{R}} \cap \mathbb{Z}^d$, suppose that $|X^a_{\mathbb{Z}}| = \infty$. Let $M = (L_1, \dots, L_s) : \mathbb{R}^d \to \mathbb{R}^s$ be a linear map such that:*

1. *The following inequalities hold, $d > 2s$ and $\mathrm{rank}(Q|_{\ker(M)}) > 2$.*
2. *The quadratic form $Q|_{\ker(M)}$ is indefinite.*
3. *For all $\alpha \in \mathbb{R}^s \setminus \{0\}$, $\alpha_1 L_1 + \dots + \alpha_s L_s$ is non rational.*

Then $\overline{M(X^a_{\mathbb{Z}})} = \mathbb{R}^s$.

For instance let α be irrational and consider

$$L(x_1, x_2, x_3, x_4) = x_1 + \alpha x_2 \quad \text{and} \quad Q(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 - x_3^2 - x_4^2$$

linear and quadratic forms respectively on $\mathbb{R}^4$. For $a = 0$, we have $|X^a_{\mathbb{Z}}| = \infty$. Moreover

$$Q|_{\ker(L)}(x_1, x_2, x_3, x_4) = \left(1 + \alpha^2\right)x_2^2 - x_3^2 - x_4^2$$

and therefore $\mathrm{rank}(Q|_{\ker(L)}) = 3 > 2$ and $Q|_{\ker(L)}$ is indefinite. It is then clear that the remaining conditions of Theorem 1.2 are satisfied for this pair, so we can conclude that the set $L(X^a_{\mathbb{Z}})$ is dense in $\mathbb{R}$.

The key feature of Theorem 1.2, that is exploited in its proof, is that $X^a_{\mathbb{R}}$ has a large group of symmetries. Moreover, there is a large subgroup, H, of this group that stabilises M and is generated by one parameter unipotent subgroups. This means that the problem