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Some results on bipartitions with 3-core

Bernard L.S. Lin

School of Sciences, Jimei University, Xiamen 361021, China

A R T I C L E I N F O

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In this paper, we investigate the arithmetic properties of bipartitions with 3-core. Let $A_3(n)$ denote the number of bipartitions with 3-core of n. We will prove one infinite family of congruences modulo 5 for $A_3(n)$. We also establish one surprising congruence modulo 14 for $A_3(8n+6)$. Finally, we prove that, if $u(n)$ denotes the number of representations of a nonnegative integer n in the form $x^2+y^2+3z^2+3t^2$ with $x, y, z, t \in \mathbb{Z}$, then $u(6n+5) = 12A_3(2n+1)$.

1. Introduction

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n. Let $p(n)$ denote the number of partitions of n. By convention, we agree that $p(0) = 1$. The generating function of $p(n)$ satisfies

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{f_1}.$$

E-mail address: linlsjmu@163.com.

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Throughout this paper, we assume that $|q| < 1$ and use the following notation:

$$f_k = (q^k; q^k)_\infty,$$

where

$$(a; q)_\infty = \prod_{n=1}^{\infty} (1 - aq^{n-1}).$$

Thanks to Ramanujan, it is well known that for $n \geqslant 0$,

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

Ramanujan's work motivates an investigation of congruence phenomena for other types of partition functions.

Given the integer partition λ of an integer n, we say that λ is a t-core if it has no hook numbers that are multiples of t. Let $a_t(n)$ denote the number of partitions of n that are t-cores. From [14, Eq. 2.1], we now know that the generating function of $a_t(n)$ equals

$$\sum_{n=0}^{\infty} a_t(n) q^n = \frac{f_t^t}{f_1}.$$

The t-cores have been studied by a number of mathematicians, see [2,3,5,9,14,15, 17–19,21,24,25], for example.

A bipartition (λ, μ) of n is a pair of partitions (λ, μ) such that the sum of all of the parts equals n. Let $p_{-2}(n)$ denote the number of bipartitions of n. The function $p_{-2}(n)$ has drawn much interest, see [1,13,12,16,26]. Recently, the arithmetic properties of bipartitions with certain restrictions on each partition have received a great deal of attention (see, for example, [6–8,10,11,20,22,23,27]).

In this paper, we shall consider arithmetic properties of bipartitions with restrictions that each partition is 3-core. A bipartition with t-core is a pair of partitions (λ, μ) such that λ and μ both are t-cores. Let $A_t(n)$ denote the number of bipartitions with t-core of n. Then the generating function of $A_t(n)$ is given by

$$\sum_{n=0}^{\infty} A_t(n) q^n = \frac{f_t^{2t}}{f_1^2}. \tag{1.1}$$

We will prove one infinite family of congruences modulo 5 for $A_3(n)$: for $\alpha \geqslant 0$ and all $n \geqslant 0$,

$$A_3\left(16^{\alpha+1} n + \frac{8 \cdot 16^\alpha - 2}{3}\right) \equiv 0 \pmod{5}. \tag{1.2}$$