Journal of Number Theory 139 (2014) 91–111

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Explicit evaluations of a level 13 analogue of the Rogers–Ramanujan continued fraction

Shaun Cooper [*], Dongxi Ye

Institute of Natural and Mathematical Sciences, Massey University-Albany, Private Bag 102904, North Shore Mail Centre, Auckland, New Zealand

A R T I C L E   I N F O

Article history:
Received 16 August 2013
Received in revised form 16 December 2013
Accepted 16 December 2013
Available online 8 February 2014
Communicated by David Goss

MSC:
primary 11J70
secondary 11B65, 11F11, 33E05

Keywords:
Kronecker limit formula
P–Q modular equation
Ramanujan–Weber class invariant
Reciprocity formula
Rogers–Ramanujan continued fraction

A B S T R A C T

The Rogers–Ramanujan continued fraction has a representation as an infinite product given by

$$q^{1/5}\prod_{j=1}^{\infty}\left(1-q^j\right)^{\left(\frac{j}{5}\right)}$$

where $|q|<1$ and $\left(\frac{j}{p}\right)$ is the Legendre symbol. In his letters to Hardy and in his notebooks, Ramanujan recorded some exact numerical values of the Rogers–Ramanujan continued fraction for specific values of q. In this work, we give explicit evaluations of the level 13 analogue defined by

$$q\prod_{j=1}^{\infty}\left(1-q^j\right)^{\left(\frac{j}{13}\right)}.$$

* Corresponding author.
E-mail addresses: s.cooper@massey.ac.nz (S. Cooper), lawrencefrommath@gmail.com (D. Ye).

http://dx.doi.org/10.1016/j.jnt.2013.12.015

1. Introduction

Throughout this paper, it is assumed that $\mathrm{Im}(\tau) > 0$ and $q = e^{2\pi i\tau}$. Let

$$\mathcal{R}(q) = \cfrac{q^{1/5}}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \cdots}}}} \tag{1.1}$$

denote the Rogers–Ramanujan continued fraction. In 1913, Ramanujan asserted in his first letter to Hardy [11, p. 29] that

$$\mathcal{R}\big(e^{-2\pi}\big) = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2},$$

$$-\mathcal{R}\big(-e^{-\pi}\big) = \sqrt{\frac{5-\sqrt{5}}{2}} - \frac{\sqrt{5}-1}{2}$$

and $\mathcal{R}(e^{-\pi\sqrt{n}})$ can be found exactly if n is a positive rational number. These results particularly impressed and intrigued Hardy who responded by writing to Ramanujan [11, p. 77]:

> *"What I should like above all is a definite proof of some of your results concerning continued fractions of the type* (1.1)*; and I am quite sure that the wisest thing you can do, in your own interests, is to let me have one as soon as possible."*

A few months later Hardy reiterated the request for a proof [11, p. 87]:

> *"If you will send me your proof written out carefully (so that it is easy to follow), I will (assuming that I agree with it—of which I have very little doubt) try to get it published for you in England. Write it in the form of a paper ... giving a full proof of the principal and most remarkable theorem, viz., that the fraction can be expressed in finite terms when* $q = e^{-\pi\sqrt{n}}$*, where* n *is rational."*

More than 25 years later Hardy recalled the profound impact that Ramanujan's evaluations of $\mathcal{R}(q)$ had had on him [17, p. 9]:

> *"(They) defeated me completely; I had never seen anything in the least like them before. A single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true because, if they were not true, no one would have had the imagination to invent them."*