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An exact degree for multivariate special polynomials

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ABSTRACT

We introduce certain special polynomials in an arbitrary number of indeterminates over a finite field. These polynomials generalize the special polynomials associated to the Goss zeta function and Goss–Dirichlet L-functions over the ring of polynomials in one indeterminate over a finite field and also capture the special values at non-positive integers of L-series associated to Drinfeld modules over Tate algebras defined over the same ring. We compute the exact degree in t_0 of these special polynomials and show that this degree is an invariant for a natural action of Goss' group of digit permutations. Finally, we characterize the vanishing of these multivariate special polynomials at $t_0 = 1$. This gives rise to a notion of trivial zeros for our polynomials generalizing that of the Goss zeta function mentioned above.

1. Introduction

1.1. Notation

Let $\mathbb{F}_q$ be the finite field with q elements of positive characteristic p, and let θ be an indeterminate. Our interest is in the ring $A := \mathbb{F}_q[\theta]$. We denote by A_+ the set of

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monic polynomials in A, and for all $d \geq 0$ we write $A_+(d)$ for those elements of A_+ whose degree in θ equals d. We write $\bar{\mathbb{F}}_q$ for the algebraic closure of $\mathbb{F}_q$, and we fix an embedding of $\bar{\mathbb{F}}_q$ into the algebraic closure of $\mathbb{F}_q(\theta)$.

We shall say a p-adic integer β is *written in base* q whenever we write $\beta = \sum_{i\geq 0} \beta_i q^i$ with $0 \leq \beta_i < q$ for all $i \geq 0$. We define the *length* of a positive integer β, written in base q as above, to be $l(\beta) := \sum_{i\geq 0} \beta_i$. Of course, note the dependence on q that we omit. Finally, for a rational number α, we let $\lfloor \alpha \rfloor \in \mathbb{Z}$ denote the greatest integer less than or equal to α.

1.2. Multivariate special polynomials

The results of this note expand upon the author's work in [13] and have been ported over from the author's dissertation [12]. We study the *multivariate special polynomials*, defined for non-negative integers $\beta_1, \dots, \beta_s$ first by the formal series

$$z(\beta_1, \dots, \beta_s, t_0) := \sum_{d\geq 0} t_0^d \sum_{a\in A_+(d)} \chi_1(a)^{\beta_1} \cdots \chi_s(a)^{\beta_s} \in \mathbb{F}_q[t_1, \dots, t_s][[t_0]].$$

Here, for all $a \in A$ and $i = 1, \dots, s$, the symbols $\chi_i(a)$ stand for the images of the maps

$$\chi_i : A \to \mathbb{F}_q[t_i] \subseteq \mathbb{F}_q[t_1, \dots, t_s]$$

determined by $\theta \mapsto t_i$. It follows from our recursive formula, Proposition 2.1, that these power series are in fact polynomials in t_0.

1.3. A comment on notation

Goss points out to us that s is traditionally the coordinate used on his "complex plane" $\mathbb{S}_\infty$ (see [5, Chapter 8] for the definition). In keeping with the established notation so far in the theory of Pellarin's multivariate L-series, we will always use s to denote a non-negative integer. As $\mathbb{S}_\infty$ does not appear in this paper, we do not expect any confusion to arise.

1.4. Drinfeld modules over Tate algebras

In [2], Anglès, Pellarin and Tavares–Ribeiro introduce the notion of Drinfeld modules over Tate algebras. The authors associate L-series to these new Drinfeld modules, generalizing the multivariate L-series studied in [1,8,9,11]. They mention in Example 4.1.3 that the recursive formula of this note (Proposition 2.1) implies that the negative special values of their L-series are finite K-linear combinations of elements that are algebraic over $\mathbb{F}_q(t_0, t_1, \dots, t_s)$. Let us briefly describe the connection.